LETTRE XVII.

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EULER à GOLDBACH.

SOMMAIRE. Recherches ultérieures sur la séparation et l'intégration de l'équation Riccati.

Domi d. 3 Januar. 1732.

Omnis aequatio ex tribus constans terminis facile reducitur ad hanc formam $x^m dx + ay^n dx + bdy = 0$, quae ista substitutione

$$x = v^{\frac{1}{mn+n-m}} z^{\frac{n-1}{mn+n-m}} \text{ et } y = v^{\frac{m+1}{mn+n-m}} z^{\frac{-1}{mn+n-m}}$$

transformatur in sequentem ordinis secundi aequationem $z^2 dv + (n-1) vzdz + avzdv + a(n-1) v^2 dz + b(m+1) zdv - bvdz = 0$. Si fuerit $n = 2$, habetur forma Riccatii

$$x^m dx + ay^2 dx + bdy = 0,$$

cui ista aequatio ordinis secundi respondet

$$z^2 dv + vzdz + avzdv + av^2 dz + b(m+1) zdv - bvdz = 0,$$

pro qua mihi difficilior videtur casuum separabilium investigatio, quam pro ipsa $x^m dx + ay^2 dx + bdy = 0$. Sit $n = 1$, erit aequatio in quam haec $x^m dx + aydx + bdy = 0$ transformatur, ista $z^2 dv + avzdv + b(m+1) zdv - bvdz = 0$, in qua littera z unicam dimensionem habere censenda est.

Quod aequationis $ady = y^2 dx - x^{\frac{-4n}{2n+1}} dx$ ad hanc $adq = q^2 dp - dp$ reductio universalis, n denotante numerum quemcunque, pendeat ab inventione termini generalis hujus seriei $\overset{m}{A}$, $\overset{m+1}{B}$, $(2m+1) \overset{m+2}{B} + A$, sic ostendo: Reductio illa perficitur hac substitutione $x = \left(\frac{p}{2n+1}\right)^{2n+1}$ et

$$y \left(\frac{p}{2n+1}\right)^{2n} = \cfrac{1}{\cfrac{(2n-1)a}{p} + \cfrac{1}{\cfrac{(2n-3)a}{p} + \cfrac{1}{\cfrac{(2n-5)a}{p} + \cfrac{1}{\text{etc. usque ad } \cfrac{1}{\cfrac{1a}{p} + q}}}}}$$

Formula ista continuarum fractionum dat, si $n = 1$, hunc valorem $\cfrac{1}{\frac{a}{p} + q}$ vel $\frac{1}{r+q}$, posito $r = \frac{a}{p}$. Si $n = 2$, prodit $\cfrac{1}{3r + \cfrac{1}{r+q}} = \frac{r+q}{3r^2 + 3rq + 1}$. Si $n = 3$, fit

$$\cfrac{1}{5r + \cfrac{1}{3r + \cfrac{1}{r+q}}} = \cfrac{1}{5r + \cfrac{r+q}{3r^2 + 3rq + 1}} = \frac{3r^2 + 3rq + 1}{15r^3 + 15r^2 q + 6r + q}.$$

Ponatur, brevitatis gratia, $r + q = s$, seu $q = s - r$ et valores inventi formulae datae respondentes litterae n collocentur in seriem, prodibit

$$n = \quad 1 \qquad 2 \qquad 3 \qquad 4$$

$$\frac{1}{s},\ \frac{s}{3rs+1},\ \frac{3rs+1}{15r^2s+5r+s},\ \frac{15r^2s+5r+s}{105r^3s+35r^2+10rs+1},\ \text{etc.}$$

in qua serie apparet cujusvis fractionis numeratorem esse praecedentis denominatorem. Atque si terminus ordine m sit $\frac{A}{B}$, fore sequentem indicis $m+1 = \frac{B}{(2m+1)B+A}$. Ex his ergo manifestum est, quod in praecedentibus litteris commemoravi, ex termino generali hujus seriei

$$\overset{m}{A},\ \overset{m+1}{B},\ \overset{m+2}{(2m+1)B+A}$$

cognito haberi formulae Riccatianae separationem et integrationem universalem. In illa autem serie, ut sit determinata, oportet esse terminum primum $= 1$ et secundum $= s$. Cognitis igitur ex termino generali A et B factoque $n = m$, erit

$$x = \left(\frac{p}{2m+1}\right)^{2m+1} \text{ et } y\left(\frac{p}{2m+1}\right)^{2m} = \frac{A}{B},$$

qua substitutione aequatio $ady = y^2dx - x^{\frac{-4m}{2m+1}}dx$ reducitur ad hanc $adq = q^2dp - dp$, ideoque integrabitur ope logarithmorum universaliter. Aequatio vero

$ady = y^2dx - x^{\frac{-4m}{2m+1}}dx$ modo initio tradito reducitur ad hanc

$$z^2dv + vzdz - vzdv - v^2dz + a\left(\frac{-2m+1}{2m+1}\right)zdv - avdz = 0.$$

Haec ergo reducetur ad istam $adq = q^2dp - dp$, substitutione $v = \frac{Ap}{(2m+1)B}$ et $z = \frac{Bp}{(2m+1)A}$. Vale et fave, V. C., Tui observantissimo

Eulero.

LETTRE XVIII.

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Goldbach à Euler.

SOMMAIRE. Remarque sur les sommes des séries et les intégrales. Solution d'une équation du 5ème degré.

Moscuae $\frac{15}{26}$ Januar. 1752.

In superioribus litteris Tuis non animadverteram Te in formula $A,\ B,\ (2m+1)\ B + A$ sumere m pro exponente terminorum qui comperit termino A, quod ex postremis Tuis nuper ad me datis nunc satis intelligo videoque simili modo $\int(1-y^{\frac{1}{n}})^p dy$ pendere a formula generali summarum seriei, cujus lex progressionis est $\big((p+n+x)\div(n\pm x)\big)A = B$, ubi per x intelligo exponentem qui termino A respondet, per $\div$ vero signum divisionis ambiguae, ita ut sumto ex signis $\pm$ superiore, $n+x$ sit denominator, sumto inferiore, $n-x$ fiat numerator; vel eandem integralem pendere a