LETTRE XV.

Euler à Goldbach.

Sommaire. Développement ultérieur du théorème précédent. Substitution pour la résolution de l'équation Riccati. Cas où $2^n - 1$ est un nombre composé, quand même n serait premier.

Petropoli die 25 Novembris A. 1751.

In integrali hujus formulae $dx \frac{(a-x)(b-x)(c-x)\text{ etc.}}{(\alpha+x)(\beta+x)(\gamma+x)\text{ etc.}}$ utique difficulter apparet, quid fiat, si litterarum α, β, γ, etc. aliquot fuerint aequales. Denominatores in aliquot integralis mei terminis tum evanescunt, et propterea ipsum integrale infinitum fieri videtur. Verum si ad signa terminorum istorum attendimus, videbuntur ii se potius destruere, atque in nihilum abire. Horum autem neutrum recte se habet; nam termini illi in infinitum crescentes junctim sumti dabunt valorem determinatum finitum quem sequenti modo inve-

stigo. Sit $\beta = \alpha$, erit difficultas in duobus integralis terminis istis

$$\frac{(\alpha+a)(\alpha+b)(\alpha+c)\text{ etc.}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\text{ etc.}}\, l\frac{x+\alpha}{\alpha} + \frac{(\beta+a)(\beta+b)(\beta+c)\text{ etc.}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)\text{ etc.}}\, l\frac{x+\beta}{\beta}$$

posita. Ad eorum verum valorem inveniendum pono $\beta = \alpha + d\alpha$, $d\alpha$ vero denotat quantitatem infinite parvam, tantumdem ergo est ac si posuissem $\beta = \alpha$. Brevitatis gratia scribo P loco $\frac{(\alpha+a)(\alpha+b)(\alpha+c)\text{ etc.}}{(\gamma-\alpha)(\delta-\alpha)\text{ etc.}}$. Sumo deinde hujus fractionis differentiale, posito tantum α variabili, sit illud $Q\,d\alpha$. Manifestum est fore $\frac{(\beta+a)(\beta+b)(\beta+c)\text{ ete.}}{(\gamma-\beta)(\delta-\beta)\text{ etc.}} = P + Q d\alpha$. Est vero etiam $\beta - \alpha = d\alpha$ et $\alpha - \beta = -d\alpha$ et $l\frac{x+\beta}{\beta} = l\frac{x+\alpha+d\alpha}{\alpha+d\alpha} = l\frac{x+\alpha}{\alpha} - \frac{x\,d\alpha}{\alpha(\alpha+x)}$. His substitutis, duo illi termini abibunt in $\frac{P}{d\alpha}l\frac{x+\alpha}{\alpha} - \frac{P}{d\alpha}l\frac{x+\alpha}{\alpha} + \frac{Px}{\alpha(\alpha+x)} - Ql\frac{x+\alpha}{\alpha} + \frac{Qxd\alpha}{\alpha(\alpha+x)}$. Horum duo priores termini sese tollunt, et postremus prae reliquis evanescit, ita ut pro valore duorum terminorum quaesito habeamus $\frac{Px}{\alpha(\alpha+x)} - Ql\frac{x+\alpha}{\alpha}$, qui in integrali eorum loco substitui debet. Est vero, ut posui, $P = \frac{(\alpha+a)(\alpha+b)(\alpha+c)\text{ etc.}}{(\gamma-\alpha)(\delta-\alpha)\text{ etc.}}$, atque ex hoc erit

$$Q = \frac{(\alpha+a)(\alpha+b)(\alpha+c)\text{ etc.}}{(\gamma-\alpha)(\delta-\alpha)\text{ etc.}}\Big(\frac{1}{\alpha+a} + \frac{1}{\alpha+b} + \frac{1}{\alpha+c} + \text{etc.} + \frac{1}{\gamma-\alpha} + \frac{1}{\delta-\alpha} + \text{etc.}\Big).$$

Notandum hic est in casu $\beta = \alpha$ non totam quantitatem esse transcendentalem, sed partem ejus esse algebraicam, cum tamen universaliter ambo termini sint transcendentes. Si jam ulterius fuerit $\gamma = \alpha$, eodem modo terminorum infinitorum valor ponendo $\gamma = \alpha + d\alpha$ determinabitur.

De formula $\int(1 - x^{\frac{1}{n}})^p dx$ non dubito quin omnes integrabilitatis casus a Te, V. C., sint eruti. Sed de reductione aequationis $(1 - x^{\frac{1}{n}})^p dx = dy$ ad hanc $dz = (p+1) z dv + n(1-z) dv : v$ dubium habeo, cum posterior aequatio nunquam sit absolute integrabilis, siquidem adjectionem constantis non negligamus. Sumamus casum simplicissimum, quo $p = 1$ et $n = 1$, erit $dz - 2z dv + \frac{z dv}{v} = \frac{dv}{v}$. Multiplicetur haec per $e^{lv - 2v}$, seu quod idem est, per $e^{-2v} v$ (e denotat hic numerum, cujus logarithmus hyperbolicus est $= 1$), prodibit $e^{-2v} v dz - 2 e^{-2v} z v dv + e^{-2v} z dv = e^{-2v} dv$, quae integrata dat $e^{-2v} v z = \text{Const.} - \frac{1}{2} e^{-2v}$ seu $2 v z + 1 = a e^{2v}$, quae algebraica non est, nisi sit $a = 0$, et propterea ea ad hanc $x - \frac{1}{2} xx = y + b$ substitutionibus algebraicis reduci non potest. Similis est ratio formulae generalis, haec enim nullo casu est integrabilis ad aequationem algebraicam, nisi constans addenda ponatur $= 0$.

Casus nuper formulae Riccatianae separabiles considerans, sequentem universalem detexi substitutionem, qua aequatio $a dq = q^2 dp - dp$ ad hanc formam $a dy = y^2 dx - x^{-\frac{4n}{2n+1}} dx$ reduci potest. Ponatur $p = (2n+1) x^{\frac{1}{2n+1}}$ atque

$$q = -\frac{a}{p} + \cfrac{1}{-\frac{3a}{p} + \cfrac{1}{-\frac{5a}{p} + \cfrac{1}{-\frac{7a}{p} + \cfrac{1}{\text{etc. etc. etc.} \; \cfrac{1}{-\frac{2(n-1)a}{p} + \cfrac{1}{x^{\frac{2n}{2n+1}} y}}}}}}$$

Haec tantum valet substitutio si n est numerus affirmativus integer, peculiarem habeo si est negativus. Quoties in hac n est numerus integer affirmativus, toties haec fractionum series abrumpitur, et quid pro q substitui debet, facile determinatur.

Reciproce etiam aequationem $a dy = y^2 dx - x^{\frac{-4n}{2n+1}} dx$ in hanc $a dq = q^2 dp - dp$ transformo hac substitutione $x = (\frac{p}{2n+1})^{2n+1}$ et $y(p : 2n+1)^{2n} =$

$$\cfrac{1}{\frac{(2n-1)a}{p} + \cfrac{1}{\frac{(2n-3)a}{p} + \cfrac{1}{\frac{(2n-5)a}{p} + \cfrac{1}{\text{etc. etc. etc.} \; \frac{3a}{p} + \cfrac{1}{\frac{a}{p} + q}}}}}$$

Facile hic cognoscitur, si valores harum continuarum fractionum inveniri possent si n denotat numeros fractos, tum formulam $a dy = y^2 dx - x^m dx$ universaliter posse construi. Interpolatio vero ista nititur inventione termini generalis pro serie hujus proprietatis, si terminus x^{mus} fuerit A, ejus sequens B debet terminus $(x+2)^{\text{mus}}$ esse $= (2m+1) B + A$, vel in numeris hujus seriei: 1, 1, 4, 21, 151, 1380, etc.

Perpendi ulterius etiam formulam $2^n - 1$, quae non potest esse numerus primus nisi sit n numerus primus, et eos investigavi casus, quibus $2^n - 1$ non est numerus primus, quamvis fuerit n talis. Exceptiones istae sunt $n = 11$, $n = 23$, $n = 83$, reliqui numeri primi omnes centenario