Leonard Euler on the Center of Similitude

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De Centro Similitudinis (E-693, *Opera Omnia* I-26, pp. 276-285) is one of Euler's late geometrical contributions. Although presented to the Petropolitan Academy of Sciences in 1777, six years before Euler's death, it did not appear in the Academy's journal until 1791.

In the past Euler has shown considerable interest in specific cases of similarity in which one figure is constructed from another in some way (consider, e.g., E-129, Opera Omnia I.27 130-180 "Investigatio curvarum quae evolutae sui similes producant", in which the objects of study are planar curves which are similar to their own evolutes). In the current work, he considers *any* two similar figures and seeks a point that "relates similarly to either figure" ("ad utramque figuram similiter referatur", Section 1).[1] Such a point he calls the *center of similitude* of the two figures. For the case of figures in the plane, Euler manages to make this notion precise and he gives a simple geometrical means of locating centers of similitude. For figures in space, Euler equivocates somewhat on what it means for two figures to be "related similarly" to a point, and this equivocation results in some flawed attempts to locate a center of similarity for similar figures in three dimensions. Nevertheless Euler's efforts, insofar as they are successful, anticipate some nineteeeth-century work in geometric transformation theory; and insofar as they fail, they provide a useful illustration, both for undergraduates and for their instructors, of the need for the notion of a *geometric transformation* in order to clarify and organize certain geometric intuitions.

Euler begins by considering two segments in the plane: a longer segment AB with endpoints A and B, and a shorter one ab, with endpoints a and b. He calls these segments "similar figures" (*figurae* similes). The points A and a he calls "homologous", whereas we might say that they "correspond." The same goes for B and b. Euler says that he is looking for a point G, which he calls the "center of similitude" of the two segments, so that triangle GAB is similar to triangle Gab. Passing over, for the moment, some obvious questions (Are segments really figures? Aren't two segments always similar?) let us consider the problem he has set himself.

Euler attacks the question in his accustomed style, verbally profuse but nonetheless clear, working backwards in such a way that the reader can easily see how he is getting his ideas. Then after presenting the solution in a formal deductive manner, he reviews it and notes how it may be reduced to something very simple and elegant. Here is Euler's final condensation.

Extend the segments AB and ab to lines. These meet at some point O. (If they don't the problem becomes very easy: the desired point G is the intersection of lines Aa and Bb.)

[1] Extended quotes from "De Centro Similitudinis" will be followed by a number in parentheses, giving the section in the paper from which the quote was drawn.

Now make a circle containing O, A and a, and another circle containing O, B and b. These circles meet at O, and at another point G, the desired center of similitude. The proof, which involves a little work with angles inscribed in circles, is left to the reader.

The significance of the point G is made clear when it is considered from the point of view of geometric transformations. Define a *similitude* as a mapping of the plane to itself, so that every segment AB is mapped onto a segment A'B' in such a way that AB/A'B' is a fixed constant, say p. (Here, the image of a point X under a transformation is denoted X'). If $p<1$, then we say that the similitude is *contractive*. Clearly, a similitude maps any triangle ABC onto a triangle A'B'C' that is similar to it. It is well known (see Coxeter [1, Chapter 5]) that every contractive similitude has a unique invariant point. Euler's center of similitude is nothing more than the invariant point of a certain contractive similitude which maps A to a and B to b. Exactly which similitude is involved will be seen shortly.

We are now in a position to see clearly what Euler had in mind when he spoke of "similar figures." Consider two similar figures P and Q (polygons or whatever) such that Q is the image of P under a contractive similarity transformation. Let G be the invariant point. If A and B are any two points of P, and a and b respectively are their images in the figure Q, then since triangle ABG is similar to triangle A'B'G' and triangle A'B'G' is just triangle abG, then triangle ABG is similar to triangle abG. In particular, a viewer located at G would perceive segment AB to occupy an angle in his field of vision equal to the angle occupied by segment ab. In other words, corresponding ("homologous") parts of the two would appear as identical in the field of vision of a person at G, if she were to turn from one figure to the next. As wholes, the figures themselves would "look the same." Although not in possession of a notion of *similitude* or *invariant point*, Euler nevertheless has managed to grasp the significance of the invariant point of a similitude. In Euler's own words:

"The problem we have dealt with up to now may be seen as concerning the science of Perspective, since, if a copy of some object is drawn accurately, it will be quite pertinent to determine that location from which, if both the object and the copy were viewed, all homologous parts would appear as contained in equal angles." (15)

There is one matter, however, that needs clarification: Euler has shown how to locate the center of similarity, but of which similitude, if any, is it the center? In the general theory of similitudes we find that any two line segments AB and ab are related by just two similtitudes. One of them is *direct*, in the sense that it maps any triangle XYZ onto a similar triangle X'Y'Z' in which the orientation X, Y and Z going around the original triangle is the same as orientation of the image points X', Y' and Z' going around the image triangle. The other similitude is *opposite*, meaning that the orientation of the image of any triangle is reversed. It turns out that Euler's geometrical construction always locates the center of the *direct* similitude that maps segment AB to segment ab.[2]

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[2] Euler's construction of the invariant point of a direct similitude may have been rediscovered later on. Coxeter gives it as an exercise ([1], p. 75) citing Casey [2] rather than Euler.

Somewhere else on the plane is a point H that is the center of the opposite similitude that maps AB to ab.

Nothing in Euler's paper indicates that he was aware of the existence of any such point. My impression is that he was unaware of "centers of opposite similitude", for, as we shall see, it seems unlikely that one who was aware of an alternate center would have approached the problem of finding centers of similitude in space in the ways that Euler did.

It will be worthwhile to locate both centers of similitude with a single construction. One way to do so is with Apollonian circles. Given two points X and Y, and a real number m>0, the set of points P such that XP/YP = m is known to form a circle, with X and Y lying on the line containing its diameter. Denote this circle by Ap(X,Y;m). Now, given segments AB and ab with AB/ab = p >1, form the circles Ap(A,a;p) and Ap(B,b;p). They will meet at G and H, the centers of direct and opposite similitude.

The definition of a similitude in space is precisely the same as for the plane: every segment XY must be mapped to a segment X'Y' so that XY/X'Y' is a fixed constant. For any two similar triangles ABC and A'B'C', with the ratio between their sides being some p>1, there are just two similitudes that map A to A', B to B' and C to C'. Again, one is "direct", and one is "opposite." We can locate these centers by considering the intersection of Apollonian spheres: the three spheres Ap(A,A';p), Ap(B,B';p) and Ap(C,C';p). Any two of these spheres meet in a circle, and the three circles meet at the two desired points. There is no unique way, starting just from a segment AB and some "homologous" segment ab, to determine a center of direct similitude, because the Apollonian spheres Ap(A,a;p) and Ap(B,b;p) meet in a circle of "centers."

Suprisingly, Euler does not appear to realize this. He takes an analytic approach to the problem of locating the center of similitude, as follows:

"Let A and a be any pair of homologous points of two similar bodies, though which the plane of the table is considered to pass; in this same plane let some side AB of the larger body be considered to lie, and let the corresponding side ab in the smaller body be located in some other plane, which intersects the plane of the table along ai, at some inclination θ…" (8)

Euler goes on to derive a system of three equations to find three unknown quantities: the perpendicular distance z from the supposed "center of similitude" G -- which he imagines to lie above the "plane of the table" -- to a point Y in this plane; the perpendicular distance y from Y to a point x on the line ai, and the distance (also called x) from x to the point a along this line. These three quantities act as coordinates for determining the location of G, starting from a, which is one of the given points in the problem. Unfortunately the three equations involve θ, which in turns depends on one's choice of the point i. Hence, as must be the case, no center of similitude G is determined uniquely from the given points A,B, a and b.

At any rate, as Euler himself points out, "these three equations are too complicated to permit the terms x, y, and z to be determined from them in practice" (11), so he has another go at the problem:

"…let us imagine an arbitrary plane passing through homologous points A and a; there will be another plane, passing through the same points A and a, in which the center of similitude G will be located, so that the intersection would be the line Aa. In order to investigate this point we should seek out two other homologous points, located outside of the [original] plane, which are located in the same plane as the line Aa, so that the four points A,a, B, and b lie on this plane." (13)

Euler appears to be working under the assumption that the planar center of similitude (the one that is determined by the four coplanar points as in the original planar case) must necessarily coincide with the center of similitude of the two three-dimensional bodies. At any rate, Euler does note that: "considerable complications would be entailed" (13) in any attempt to locate four such coplanar points for any given pair of similar figures. To get around this difficulty, at least, Euler suggests a trick:

"Nevertheless, we may avoid such complications if we take the point B so that it lies on the line Aa itself" (14).

The idea is that, given two similar bodies, it should not be difficult to locate a point A in the larger body which is such that a line segment drawn from A to the homologous point a in the smaller body passes through some point B in the larger body. The four points -- A, B, a and the point b in the smaller body that corresponds to B -- would lie on a single plane, and the center of similitude of the two bodies could be found by applying the solution for the planar case to the coplanar segments AB and ab:

"Therefore the desired center of similitude must necessarily be located in the same plane, and this center will be found most promptly, and without any difficulty, by application of the method described in the beginning." (14)

By now we know that, in general, this method cannot work. What emerges will only be one of the points on the intersection of the spheres Ap(A,a;p) and Ap(B,b;p). Such a point is not guaranteed to be either one of the two centers of similitude of the bodies in space, for it is not generally true that, if T is a similitude in space and A and B are points such that A,B,A' and B' are coplanar, then the invariant point of T must lie on the plane determined by A, B, A' and B'.[3] There is a special case, however, in which the plane containing these four points also contains both centers of similitude. It is known (Coxeter, Chapter 7) that every contractive similitude is a *dilative rotation*:, that is: a rotation about a fixed line l, followed by a contraction about a fixed point P on l. (If the similitude is *opposite*, then the contraction carries a point X to a point X' so that P lies

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[3] Consider the similitude T of space, obtained by a 180 degree rotation about the z-axis, followed by a contraction of factor ½ toward the origin. The invariant point of T is the origin. Consider the points (-1,1,1) and (-1,-1,1), along with their respective images under T: (1/2,-1/2,1/2) and (1/2,1/2,1/2). Those four points are coplanar, but the plane does not contain the origin.

on the segment XX' between X and X'. If the similarity is *direct*, X' is between P and X.) Clearly, P must be the invariant point: the center of this similitude. It is also easy to see that a contractive similitude has an invariant plane, namely: the plane containing P that is normal to l. If the point A chosen initially by Euler lies on the invariant plane, then so must its image a; hence B, lying on the segment Aa, lies also in this plane, and therefore so does b. Applying Euler's planar method to these four points on the invariant plane locates the center of similitude P.

Euler has another approach that, although fundamentally flawed in the same way as previous attempts, is nevertheless worth a look. Consider first a planar problem, a variant of the one with which Euler began his paper: given points A, B, and C in the plane, can we locate a point O so that triangle OAB is similar to triangle OBC? (Such a point he also calls a center of similitude.) Once again, from some initial explorations, Euler distills an elegant solution:

"From the three given points A, B, and C let segments be drawn that are normal to AB and BC and whose intersections give two points E and F; then draw EF and let the perpendicular BO be dropped to it from B, and the point O will be the desired center of similitude …". (22)

The proof that, OAB and OBC are similar, is left to the reader. (Hint: AEOB and BOFC are cyclic quadrilaterals.)

Euler applies the foregoing construction to the three-dimensional problem as follows:

"Let us now consider two homologous points A and a, of which the first point A is taken to be in the object itself, and the other is taken to be in its image [Euler is not using the term "image" in the sense of "image of a set under a certain function"; he simply imagines that the smaller body is a faithfully-drawn copy of the original body.]; then, just as a segment Aa may be seen extending to the [smaller, image] object itself, so also in this manner a similar segment a α may be drawn from the point a. This segment is understood to relate to the image in the same way as the line Aa relates to the [larger, original] object itself. From what was done in the beginning, it should be maintained that the center of similitude is located in the plane that is determined by these two segments Aa and aα …" (16)

Euler's strategy may be restated, using transformations, as follows: given any point A in the original body, let a be the its image under some contractive similarity transformation: thus a is the corresponding point in the image, under the similitude, of the original body. Now apply the similitude again, getting a third body (the image of the image of the original), and denote the image of a by α. The segments Aa and aα are coplanar, and furthermore the point O in this plane, that makes OAa similar to Oaα, is the center of a (direct) similitude of the plane that takes A to a and takes a to α. Hence O must be the center of similitude for the bodies in space.

The flaw remains: in order for the point O to be the center of (direct) similitude of the bodies in space, the initial point A must lie on the invariant plane of the similitude.

By now it should be clear that the lack of an explicit notion of similarity transformation makes it very difficult to understand just what a "center of similitude" should be in the case of bodies in space. Euler himself seems to have experienced this difficulty: although he clearly states that, from the center of similitude, corresponding parts of two similar bodies should appear as contained in equal angles, he also seems to want something more from a such a point. In one place he refers to a center of similitude as a point such that "…all sections of the two bodies, formed [by the intersection of the bodies with the plane] through the two homologous points [an arbitrary point A in the original body, and it image a] and the center of similitude, are always similar figures." In general, however, there need not be such a point. Euler's apparently believed that such a point exists and that it coincided with the center from which the two objects are seen as contained in equal angles. This mistaken belief probably motivated his unsuccessful attempts to reduce the spatial problem to a simpler problem involving planar cross-sections.

Nevertheless Euler's paper anticipates later work on invariant points of geometric transformations. It may even be the case, as was asserted by J. Tropfke (*Geschichte der Elementarmathematik*, cited in [3]), that *De Centro Similitudinis* inspired other mathematicians to develop transformation theory itself. At the very least, the very simple and elegant planar constructions devised by Euler in *De Centro Similitudinis* give ample confirmation to the remark, attributed to Lagrange [4], that "…if one wanted to be a geometer, it was essential to study Euler."

References:

[1] H.M.S. Coxeter, *Introduction to Geometry,* 2nd Edition, John Wiley and Sons, 1969.

[2] J. Casey, *A Sequel to the First Six Books of the Elements of Euclid,* 6th Edition, Hodges Figgis, Dublin, 1892.

[4] Julian P. Coolidge, *A History of Geometrical Methods,* Oxford University Press, 1940.

[4] Ivor Gratten-Guiness, "On the influence of Euler's mathematics in France during the period 1795-1825", in *Festtakt und Wissenschaffliche Konferenz aus Anlass des 200-Todestages von Leonard Euler (1983),* Akademie-Verlag, Berlin.