PROOF OF SOME NOTABLE PROPERTIES WITH WHICH SOLIDS ENCLOSED BY PLANE FACES ARE ENDOWED[1]

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(Translated by Christopher Francese[2] and David Richeson[3])

Just as plane rectilinear figures, whose nature is commonly investigated in Geometry, have certain well known general properties, such as that the number of angles is equal to the number of sides and that the sum of the angles is equal to a number of right angles which is four less than twice the number of sides, so have I recently outlined the first principles of a Solid Geometry of the same type, including similar properties belonging to solids enclosed by plane faces. In Solid Geometry those bodies which are bounded on all sides by plane faces rightly merit first consideration, just as rectilinear figures do in Planar Geometry, or what is properly called Geometry. I have decided to establish similar principles of Solid Geometry which govern the formation of solids and on the basis of which their properties can especially be proved. In this matter it is quite surprising that although Solid Geometry has been studied for as many centuries as Geometry, its first principles are practically unknown. Nor has anyone, in such a long time, attempted to investigate this subject and put it into order. And so I undertook this task. But, although I had uncovered many properties which are common to all bodies enclosed by plane faces and which seemed to be completely analogous to those which are commonly included among the first principles of rectilinear plane figures, still, not without a great deal of surprise did I realize that the most important of those principles were so recondite that all the time and effort spent looking for a proof of them had been fruitless. Nor, when I consulted my friends, who are otherwise extremely versed in these matters and with whom I had

Date: November 22, 2004.

[1]Translation of "Demonstratio nonnullarum insignium proprieatatum, quibus solida hedris planis inclusa sunt praedita," *Novi Commentarii Academiae Scientiarum Petropolitanae*, 4:72-93, 1758

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shared those properties, were they able to shed any light from which I could derive these missing proofs. After the consideration of many types of solids I came to the point where I understood that the properties which I had perceived in them clearly extended to all solids, even if it was not possible for me to show this in a rigorous proof. Thus, I thought that those properties should be included in that class of truths which we can, at any rate, acknowledge, but which it is not possible to prove.

However, the general properties of solids, which still require proof, depend upon one property in such a way that if it were possible to prove this property, then all of the first principles of Solid Geometry which I have proposed would be equally as firm as the first principles of Geometry. So that property, not yet proved, which contains so many properties in itself, is found in the following proposition:

In every solid enclosed by plane faces the number of solid angles, along with the number of faces, exceeds the number of edges by two.

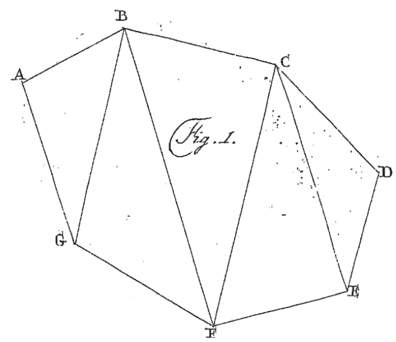
From this I have derived another equally notable property common to all solids of this type, which is stated as follows:

In every solid enclosed by plane faces the sum of all of the plane angles which make up the solid angles is equal to a number of right angles which is eight less than four times the number solid angles.

This proposition is so connected to the previous one that if the one can be proved, then the proof of the other is obtained at the same time; hence, a deficiency in the first principles of Solid Geometry that I have published will be remedied if a proof of either of these two propositions is found.

When I had considered this proposition anew, I finally found the desired proofs of these propositions. I arrived at proofs similar to the one customarily used for the analogous proposition from Geometry regarding the sum of the angles of any rectilinear figure. In Geometry any rectilinear figure can be ultimately reduced to a triangle by successive division of angles. Likewise, given any solid enclosed by plane faces, I observed that the solid angles can be continuously divided so that, finally, a triangular pyramid remains. Since a triangular pyramid is the most simple figure among solids, I perceived that on the basis of its known properties one could generalize to the properties of all solids. For in any triangular pyramid the number of solid angles is four, the number of faces is four and the number of edges is six, whose double, twelve, gives the number of plane angles, whose sum is equal to eight right angles.

If straight lines are drawn from any point within a solid to each solid angle, then the solid will be divided into as many pyramids as there are faces, in as much as each face will form the base of a pyramid, while their vertices meet at the point. These pyramids, if they are not triangular, will be quite easily dissected into triangular pyramids. But this method of dividing any solid into triangular pyramids is not relevant to the present inquiry. Here I will explain a second method by which any solid is reduced by successive cutting of its solid angles to triangular pyramids. From here, finally, a proof of the principles I have mentioned will easily be obtained.



This operation is similar to that by which any rectilinear figure is customarily reduced to a triangle by successive cuttings of its angles. For if we have a plane figure with sides $ABCDEFGA$ (Fig. 1), and if the triangle CDE is cut from it by the line CE, the figure that remains is $ABCEFGA$, whose number of angles will be less by one. Now if, again, the triangle CFE is cut by the line CF the figure $ABCFGA$ will remain. If, from this, we next remove the triangle BCF and then the triangle BGF, finally the triangle ABG will remain.

From this division both outstanding properties of plane figures are easily proved. Let the number of sides of figure $ABCDEFG$ be equal to L and the number of its angles be equal to A. If angle D is cut off by a straight line CE, the number of angles in the remaining figure will be $A-1$, and, because the two sides CD and DE have been removed but in their place a new side CE has been added, the number of sides will equal $L-1$. From this it is clear that if one angle is cut off again the number of angles will equal $A-2$ and the number of sides will equal $L-2$. If now, in this way, n angles are cut off, the number of angles in the remaining figure will equal $A-n$ and the number of sides will equal $L-n$. When this remaining figure is a triangle, $A-n=3$ and $L-n=3$, from which it follows that $L=A$. That is, in any rectilinear figure the number of sides is equal to the number of angles.

Next let R be the number of right angles to which all of the angles of the proposed figure $ABCDEFG$, taken together, are equal. With the removal of angle D, that is, triangle CDE, the three angles of the triangle CDE would be removed from the angles of the figure. Since the removed angles are equal to two right angles the sum of the angles of the remaining figure $ABCEFG$ will equal $R-2$ right angles. The remaining number of angles now equals $A-1$. If an angle is cut away again so that the number of angles is equal to $A-2$, their sum will equal $R-4$ right angles. And if, now, we remove n angles the number of the angles in the remaining figure will be $A-n$ and their sum would equal $R-2n$ right angles. When that remaining figure is now a triangle, in other words $A-n=3$, since the sum of the angles equals two right angles, $R-2n=2$. Thus $2A-2n=6$, and if, from this, we subtract the former equation, then $2A-R=4$. In other words, $R=2A-4=2L-4$. So it is established that in any polygon the sum of all of the angles is equal to a number of right angles which is four less than twice the number of sides.

Therefore, in the same manner in which I have elicited from such a cutting of rectilinear figures the two crucial properties of figures of this type, I shall initiate an investigation for solids. By successive cuttings of solid angles I shall reduce all solids enclosed by plane faces, finally, to triangular pyramids. When I arrive at that point the number of solid angles, the number of faces, the number of edges and the sum of all the plane angles will be known. So that these things might be clearer I will present the entire matter in the following propositions.

PROPOSITION 1. PROBLEM

1. *Given a solid enclosed everywhere by plane faces, cut a given solid angle from it in such a way that in the resulting solid the number of solid angles is lesser by one.*

SOLUTION

Let O (Fig. 2) be the solid angle to be cut off, where the edges AO, BO, CO, DO, EO, FO meet in such a way that O is formed by the plane angles AOB, BOC, COD, DOE, EOF, FOE, and points A, B, C, D, E, F represent the adjacent solid angles of the body which are connected to O by straight lines AO, BO, CO, DO, EO, FO. Now, a part must be separated from the solid in such a way that solid angle O is completely removed but everything else remains, without, however, forming a new solid angle. So, the first cutting should be made through an adjacent angle, B, along the plane ABC until it reaches angles A and C; then let there be a cutting beginning at O along AOC. In this way the triangular pyramid $OABC$ will be cut away from the solid. Then, by applying the knife to AC, let a cutting be directed to angle F through the plane AFC, and from O let there

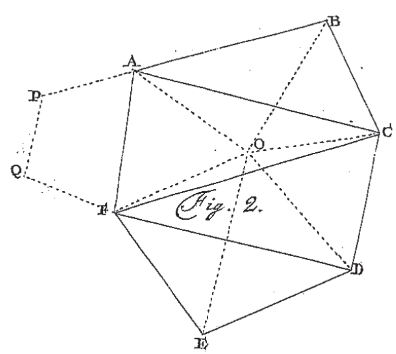


Fig. 2.

be another cutting along FOC so that the triangular pyramid $OACF$ is separated. Next, let the solid be cut along the plane CDF and let another cutting from O be made to DF with the result that in this way the triangular pyramid $OCDF$ is cut away. Finally, a cutting made along DEF will cut away the triangular pyramid $ODEF$. And thus the solid angle O will have been completely cut off. Because the rest of the solid angles remain and no new solid angle has been formed by the cuttings, the number of solid angles in the resulting solid will be diminished by one. Q[uod] E[rat] F[aciendum]

COROLLARY 1

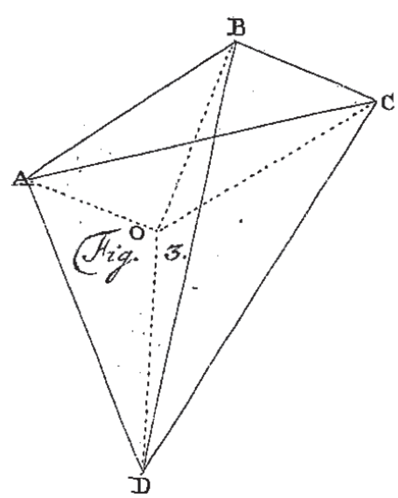
2. If the solid itself is a triangular pyramid it will be completely removed by a cutting of this type so that nothing is left. But, because we have begun this cutting in order to reduce the solid to a triangular pyramid in the end, if it is already a pyramid of this type there will clearly be no need for cutting.

COROLLARY 2

3. If a solid angle O which is to be cut away from the solid is formed from only three plane angles, i.e., if only three edges come together in it, then it will be separated from the solid by a single cutting and in this way a single triangular pyramid will be removed.

COROLLARY 3

4. If solid angle O is formed by four plane angles and the same number of edges come together in it, then two triangular pyramids must be cut off in order to remove it. This can be done in two ways (Fig. 3): two pyramids will have to be cut away, either $OABC$ and $OACD$ or $OABD$ and $OBCD$. And if points A, B, C, D are not in the same plane the resulting solids will have a different shape accordingly.



COROLLARY 4

5. If the solid angle is formed from five plane angles, and the straight lines which meet at it are extended to five other solid angles, then angle O will be cut away by separating three triangular pyramids. This can be accomplished in five different ways which also yield different results unless the five adjacent solid angles are situated in the same plane.

COROLLARY 5

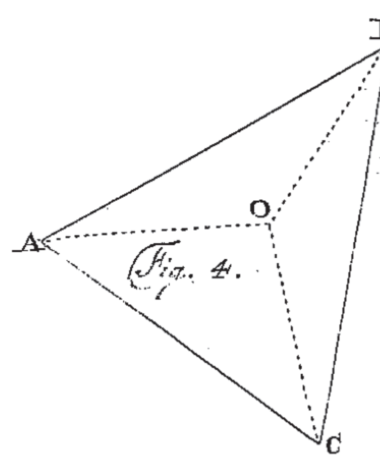
6. The cutting off of one solid angle can be undertaken at any angle of the proposed solid, and, unless only three plane angles come together to form the solid angle, that cutting can be undertaken in many ways. Therefore, it is clear that unless it is already a triangular pyramid, any solid body can be shorn of one solid angle in many ways.

COROLLARY 6

7. Thus, no matter how many solid angles the proposed body possesses, provided that the number is continually diminished by one in this way, eventually, when only four solid angles remain, it will have been reduced to a triangular pyramid. Because each part that has been removed is a triangular pyramid, by this method the entire solid will be divided into triangular pyramids.

SCHOLION

8. If the number of solid angles in the proposed body equals S, then, after one of them has been cut away in the manner indicated, the number of solid angles in the resulting body will be $S - 1$. Since the force of the proposition is contained in this diminution it seems to require qualification in many instances. If the proposed solid is a triangular pyramid (Fig. 4), then when one angle is cut away the whole pyramid is removed at the same time such that nothing remains. For, when a cutting is made along the plane ABC,



which constitutes the base of the pyramid $OABC$, the entire pyramid is cut away at the same time. But in this case the matter can be conceived as if the base ABC were left behind. Even though it is a plane figure endowed with no thickness, it can be regarded as the image of a solid consisting of only three angles, and which must be thought of as having two faces and three edges. It will render, as it were, a triangular prism of vanishing height in which the lateral faces diminish to nothing and the upper base, with its angles, falls into the lower base. In this way, both of the above mentioned properties of solids remain in force. Because the number of solid angles in this case would be $S = 3$, the number of faces $H = 2$, and the number of edges $A = 3$, it is clear that $S + H = A + 2$. Moreover, the sum of the plane angles contained in each face is equal to four right angles, which number equals $4S - 8$. The same thing happens in all pyramids. If the vertical angle O is cut away from it when the whole pyramid is removed at the same time, then the base alone should be conceived of as remaining. If it is a polygon of n sides, it will be able to be looked at as the image of a solid in which the number of solid angles would be $S = n$, the number of faces $H = 2$, and the number of edges $A = n$ such that, once again, $S + H = A + 2$. Furthermore, since each face is a polygon of n sides all the angles contained in both will equal $4n - 8 = 4S - 8$ right angles, just as the second Theorem postulates. However, even if these cases do not oppose the truth, still, in the present discussion, there is no need to pay attention to them. Since it has been proposed to reduce all solids to triangular pyramids, if the solid were already a pyramid of that type, there would be no need whatsoever for the removal of any angles. If it should be a pyramid having a base of many sides, then it will be convenient to cut off not the vertex angle, but one of the angles situated on the base which are formed by only three plane angles. In this way, after the cutting, a pyramid will always remain whose number of solid angles will be one less than previously. In general, whatever solid is proposed, it will always be suitable for the cutting to begin at the solid angle which is formed from as few plane angles as possible so that some portion of the solid remains, until it arrives at the shape of a triangular pyramid. Meanwhile, however, the force of the following proofs does not depend on this requirement, and so I have simply added it at the end of the discussion so that an apparent difficulty will nevertheless not be ignored.

PROPOSITION 2. PROBLEM

9. *If any solid angle is removed from the proposed body in the manner previously explained, and in this way the number of solid angles is diminished by one, determine both the number of faces and the number of edges in the remaining solid, and likewise determine the sum of all the plane angles.*

SOLUTION

For the proposed solid, let the number of solid angles equal S, the number of faces equal H, the number of edges equal A and the sum of all of the plane angles equal R right angles. Now, let the solid angle O (Fig. 2) be cut away in such a way that when it has been cut away, the number of solid angles in the remaining solid will equal $S-1$. So that we might understand the remaining dispositions of the resulting solid let us consider first the sum of the plane angles, which in the original solid we posited as equalling R right angles. First of all, with the cutting away of angle O, all of the angles contained in triangles AOB, BOC, COD, DOE, EOF, and FOA are subtracted from the computation of the plane angles since these triangles are separated from the surface of the body. Let n be the number of these triangles, or of the adjacent angles A, B, C, D, etc. The sum of the subtracted angles will be $2n$ right angles. But, when these triangles have been removed, the face of the solid in their place will now be bounded by triangles ABC, ACF, CFD, and DFE whose number is smaller than that number by two, and so, is $n-2$. When the angles of these triangles whose sum is $2n-4$ right angles, are added, it is manifest that through the cutting away of solid angle O the sum of the plane angles R is diminished at first by two $2n$ right angles, then, however, it is increased again by $2n-4$ right angles. So, the total diminution will be 4 right angles. Thus, in the resulting solid the sum of all the plane angles will be equal to $R-4$ right angles. In this way, whatever solid angle is cut away, the sum of all the plane angles is diminished by 4 right angles.

If all the faces which come together at O are triangles, the cutting off of the angle O means that all of those faces are removed. If the number of those faces is said to be n, the number of faces H will be diminished by n. But, in place of these faces new triangular faces which arose in the cutting will appear on the surface of the solid, that is, ABC, ACF, CFD, and DFE, whose number number is $n-2$. Thus the number of faces, which was previously H, will now be

$$H-n+(n-2)=H-2.$$

But, if it should happen that two or more of these triangles are situated in the same plane, such as if the triangles ABC and ACF are positioned in the same plane, they now will be reckoned to exhibit not two, but a single quadrilateral face such that the number of faces will be $H-3$. If it happens that the triangles of two faces of this type fall into the same plane μ times, the number of faces will equal $H-2-\mu$. But, if not all of the faces which come together at O are triangular, but one, for example $AOFQP$, consists of many sides, it is clear that by the cutting away of triangle AOF the face is not entirely removed, but the remaining part $AFQP$ still enters into the count of faces. Thus, the number of faces will be $H - 2 - \mu + 1$. If among the faces which come together at O there are found ν nontriangular faces, the number of faces remaining will be $H - 2 - \mu + \nu$. As for the number of edges which will be left after the cutting away of angle O, let us suppose for the beginning of our investigation, as previously, that all the faces which come together at O are triangular. First of all, edges OA, OB, OC, OD, etc., whose number equals n, will be subtracted from the number of edges. But in their place the new edges AC, CF, FD, whose number equals $n - 3$, will be added. Thus the number of edges will be

$$A - n + (n - 3) = A - 3,$$

if, in fact, the new faces ABC, ACF, etc. are inclined with respect to each other. But if two of them, ABC and ACF, are situated in the same plane with the result that they are reckoned to constitute a single face, edge AC will disappear and the number of edges will be $A - 3 - 1$. If it happens that the triangles of two such faces fall upon the same plane μ times, as we have described earlier, the number of edges will be $A - 3 - \mu$. Moreover, if any of the faces which form the angle O is not triangular, for example face $AOFQP$, then with the separation of triangle AOF a new edge appears, AF, which was not present before, whence the number of edges, in this case, is increased by one. But if, as we have explained previously, among the faces coming together at O, ν faces are found to be nontriangular, the number of edges in the proposed solid after the removal of the angle O will be $A - 3 - \mu + \nu$ although previously it had been equal to A. Q[uod] E[rat] I[nvestigandum]

COROLLARY 1

10. So, if a solid enclosed by plane faces is shorn by one solid angle with the result that the number of solid angles is now equal to $S - 1$, while previously it was equal to S, the sum of all of the plane angles is diminished by four right angles. In other words, while it had been previously equal to R right angles, now it will be equal to $R - 4$ right angles.

COROLLARY 2

11. Since the number of faces, which was previously equal to H, now after the cutting off of angle O is equal to $H - 2 - \mu + \nu$, it is clearly possible that the number of faces would turn out to be greater. This will happen if $\nu > 2 + \mu$ where μ and ν have those values which were

assigned in the solution.

COROLLARY 3

12. It is clear that the same thing can happen in the number edges, which, while before the removal of the angle O was equal to A, now has been found to equal $A - 3 - \mu + \nu$. This number is greater than the former total if $\nu > 3 + \mu$. Therefore, in this case, the number of faces increases even more.

COROLLARY 4

13. Since, in the expressions $H - 2 - \mu + \nu$ and $A - 3 - \mu + \nu$, the letters μ and ν signify the same thing, it is clear that the decrease in the number of edges A is greater by one than the decrease in the number of faces. Thus, after the cutting away of one solid angle, if the number of faces becomes equal to $H - \alpha$, the number of edges will become equal to $A - \alpha - 1$.

COROLLARY 5

14. Therefore, it follows that the difference between the number of faces and the number of edges, which in the beginning was equal to $A - H$, now after the removal of one solid angle will be equal to $A - H - 1$. Of course, in whatever way the resulting solid is composed, this difference always becomes smaller by one through the computation of the variables μ and ν.

SCHOLION

15. From the preceding it will now be possible, very easily, to obtain proofs of the Theorems mentioned above. These proofs are in no way inferior to those proofs used in Geometry except that here due to the nature of solids one must use more imagination, in as much as solids are being depicted on a flat surface. But if corporeal figures of this type were to be fashioned, all things would be just as clear. But the things which I have assumed in the solution of that problem are clear in and of themselves, for if you take a polygon $ABCDEF$ bounded by n sides it will quickly become apparent to one paying only slight attention that if that figure is dissected into triangles with diagonal lines, the number of those triangles will be $n - 2$ and the number of diagonals drawn in this way will equal $n - 3$. A quadrilateral is divided by one diagonal into two triangles, a pentagon is divided by two diagonals into three triangles, a hexagon is divided by three diagonals into four triangles,

and so on.

PROPOSITION 3. THEOREM

16. *In every solid enclosed by plane faces, the sum of all the plane angles which exist in its faces is equal to a number of right angles which is four times the number of solid angles minus eight; that is, if the number of solid angles is equal to* S*, the sum of all of the plane angles is equal to* $4S-8$ *right angles.*

PROOF

In any solid, let the number of solid angles be equal to S and the sum of all of the plane angles be equal to R right angles. So it is to be proved that $R = 4S - 8$. Now, in the method previously indicated, let a single solid angle be separated from the solid, so that the number of solid angles which it will have is equal to $S-1$, and the sum of the plane angles will be equal to $R-4$ right angles. If a solid angle will again be cut away so that the number remaining is $S-2$, the sum of the plane angles will be equal to $R-8$. By continuing in this way, it will be clear, for any number of solid angles, what the sum of all the plane angles will be, as the following table indicates.

Number of solid angles	Sum of all the plane angles
S	R
$S-1$	$R-4$
$S-2$	$R-8$
$S-3$	$R-12$
$\vdots$	$\vdots$
$S-n$	$R-4n$

Therefore, when, by this continual shearing away of pieces, we have arrived at $S-n$ solid angles, the sum of the plane angles will be equal to $R-4n$ right angles. But in this way we shall finally arrive at four solid angles, in which case the solid will take the form of a triangular pyramid, in which it is agreed that the sum of all of the plane angles is equal to 8 right angles. That is, if $S-n=4$, then $R-4n=8$, or $R = 4n+8$. Hence, $n = S-4$, and when this value is substituted here we get

$$R = 4S - 16 + 8 = 4S - 8.$$

So, in any solid the sum of the plane angles will equal a number of right angles which is four times the number of solid angles minus eight. Q.E.D.

SCHOLION

17. A second theorem depends on this one in such away that when this one has been proved, the truth of the other is vindicated at the same time. Nevertheless, on the basis of the forgoing problem a proof of the other theorem as well can be produced in the following manner.

PROPOSITION 4. THEOREM

18. *In every solid enclosed by plane faces, the number of faces along with the number of solid angles exceeds the number of edges by two.*

PROOF

Let it be that in any proposed solid:

Number of solid angles	=	S,
Number of faces	=	H,
Number of edges	=	A.

As we saw before, if, by cutting away one solid angle, the number S is diminished by 1, with the result that it is $S-1$, then the difference between the number of edges and the number of faces will be equal to $A-H-1$. Therefore by continuing this process of shearing off,

if the number of solid angles is	the excess of the number of edges over and above the number of faces is
S	$A-H$
$S-1$	$A-H-1$
$S-2$	$A-H-2$
$S-3$	$A-H-3$
$\vdots$	$\vdots$
$S-n$	$A-H-n$

Therefore, in this way we will arrive at a triangular pyramid in which the number of solid angles is equal to 4, the number of faces is equal to 4, and the number of edges is equal to 6, so that the excess of the number of edges over and above the number of faces will be equal to two. It is evident that if $S-n=4$, then $A-H-n=2$. From which it follows that $n=S-4$, and thus $n=A-H-2$. Thus it holds that

$$S-4=A-H-2, \text{ or, } H+S=A+2.$$

From this it is clear that for every solid enclosed by plane faces the number of faces, H, along with the number of solid angles, S, exceeds the number of edges, A, by two. Q.E.D.

SCHOLION

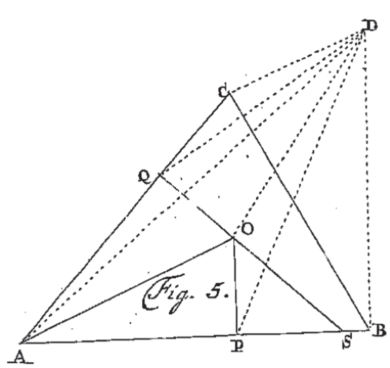
19. Now that these Theorems have been proved the first principles of Solid Geometry, which I explained some time ago, have been fortified with very sound proofs, so that they yield not a bit to the first principles of Geometry. However, I admit that I have thus brought to light only the first principles of Solid Geometry, on which this science should be built as it develops further. No doubt it contains many outstanding qualities of solids of which we are so far completely ignorant. However, since the volume of any proposed solid is accustomed to be sought after, I shall provide, as an appendix, a method for finding the volume of any triangular pyramid. For since when any point is taken on the inside of a solid enclosed by plane faces, the solid is resolved into as many pyramids as it has faces such that every face forms the base of a pyramid, and any pyramid whose base is not triangular is easily resolved into triangular pyramids, it is sufficient to have found the volume a triangular pyramid. Since this is true, if the base is multiplied by one third of the altitude, I will show how when the sides of the pyramid have been given, from those the volume can be determined, just as the area of a triangle is accustomed to be determined from three given sides.

PROPOSITION 5. PROBLEM

20. *Given six sides or edges of a triangular pyramid, find its volume.*

SOLUTION

Let there be a triangular pyramid $ABCD$ (Fig. 5) whose base is the triangle ABC and vertex D, and let it have sides: $AB = a$, $AC = b$, $BC = c$, $AD = d$, $BD = e$, and $CD = f$. Now, in faces ADB and ADC let perpendicular lines DP and DQ be sent from D to the bases opposite, and in base ABC from points P and Q let there be led toward sides AB and AC normal lines PO and QO intersecting at O. DO will be a straight perpendicular line from vertex D to base ABC, whence the volume of the pyramid will be $\frac{1}{3}DO \times$ area ABC; and when AO has been drawn



$$DO = \sqrt{AD^2 - AO^2} = \sqrt{AD^2 - AP^2 - PO^2}.$$

Now, from the basic principles of geometry it is clear that

$$AP = \frac{aa+dd-ee}{2a} \text{ and } AQ = \frac{bb+dd-ff}{2b}.$$

From here when QO is extended into S if the angle BAC is called α, then

$$QS = AQ\tan\alpha \text{ and } AS = \frac{AQ}{\cos\alpha},$$

thus

$$PS = \frac{AQ}{\cos\alpha} - AP.$$

Since $QS : AQ : AS = PS : PO : OS$,

$$PO = \frac{AQ\cdot PS}{QS} = \frac{PS}{\tan\alpha} = \frac{AQ}{\sin\alpha} - \frac{AP}{\tan\alpha}, \text{ that is, } PO = \frac{AQ - AP\cos\alpha}{\sin\alpha};$$

then, indeed

$$OS = \frac{AS\cdot PS}{QS} = \frac{PS}{\sin\alpha} = \frac{AQ}{\sin\alpha\cos\alpha} - \frac{AP}{\sin\alpha}$$

in the same way

$$QO = QS - OS = AQ\tan\alpha - \frac{AQ}{\sin\alpha\cos\alpha} + \frac{AP}{\sin\alpha} = \frac{AP - AQ\cos\alpha}{\sin\alpha}.$$

Hence it will be that

$$AO^2 = AP^2 + PO^2 = \frac{AP^2 + AQ^2 - 2AP\cdot AQ\cos\alpha}{\sin^2\alpha};$$

In the same way

$$DO^2 = \frac{AD^2\sin^2\alpha - AP^2 - AQ^2 + 2AP\cdot AQ\cos\alpha}{\sin^2\alpha}.$$

But the area of triangle ABC is equal to $\frac{1}{2}ab\sin\alpha$, from which the volume of the pyramid equals

$$\frac{1}{6}ab\sqrt{AD^2\sin^2\alpha - AP^2 - AQ^2 + 2AP\cdot AQ\cos\alpha}$$

$$= \frac{1}{6}\sqrt{\begin{pmatrix} aabbdd\sin^2\alpha - \frac{1}{4}bb(aa+dd-ee)^2 - \frac{1}{4}aa(bb+dd-ff)^2 \\ +\frac{1}{2}ab(aa+dd-ee)(bb+dd-ff)\cos\alpha \end{pmatrix}}.$$

Finally, from triangle ABC it is the case that

$$\cos\alpha = \frac{aa+bb-cc}{2ab} \text{ and in the same way } \sin^2\alpha = 1 - \frac{1}{4aabb}(aa+bb-cc)^2,$$

and when these values are substituted the volume of the pyramid will result:

$$\frac{1}{12}\sqrt{\left(\begin{array}{c} 4aabbdd - dd(aa+bb-cc)^2 - bb(aa+dd-ee)^2 - aa(bb+dd-ff)^2 \\ +(aa+bb-cc)(aa+dd-ee)(bb+dd-ff) \end{array}\right)},$$

which, when the terms are expanded, can be expressed in the following form

$$\frac{1}{12}\sqrt{\left(\begin{array}{c} aaccdd + aabbee + aabbff + aaddff + bbccdd + bbddee \\ aaccff + aaeeff + bbccee + bbeeff + ccddee + ccddff \\ -aabbcc - aaddee - bbddff - cceeff \\ a^4ff - aaf^4 - b^4ee - bbe^4 - c^4dd - ccd^4 \end{array}\right)},$$

which seems to be able to displayed still more conveniently as follows

$$\frac{1}{12}\sqrt{\left(\begin{array}{c} aaff(bb+cc+dd+ee) - aaff(aa+ff) - aabbcc \\ +bbee(aa+cc+dd+ff) - bbee(bb+ee) - aaddee \\ +ccdd(aa+bb+ee+ff) - ccdd(cc+dd) - bbddff - cceeff \end{array}\right)}.$$

And thus from the six given sides a, b, c, d, e, f of the triangular pyramid its volume is determined. Q.E.I.

SCHOLION 1

21. So that the method may be more clearly seen by which, in this expression, the sides a, b, c, d, e, f are combined, it should be noted that four triangles are formed from them. That is,

$\triangle ABC$ consists of sides a, b, c,

$\triangle ABD$ consists of sides a, d, e,

$\triangle ACD$ consists of sides b, d, f,

$\triangle BCD$ consists of sides c, e, f.

Whence it is clear that side a comes together with each of the remaining sides to form triangles, except with side f, for which reason I shall call these sides a and f disjoint, because they are not joined to each other. In the same way sides b and e will be disjoint, likewise sides c and d.

Therefore, immediately after the radical sign there occur terms formed from the disjoint sides $aaff$, $bbee$, $ccdd$, which are multiplied by the sum of the remaining squares. Then the same terms, taken negatively, are multiplied by the sum of their own squares, and from this, finally, are subtracted the results from the squares of the three sides of each triangle.

SCHOLION 2

22. A somewhat simpler formula for the volume of a pyramid can also be found if there are given only three sides converging on one solid angle, along with the plane angles which they form there.

Let there be three sides coming together in a solid angle A,

$$AB = a,\ AC = b,\ AD = d,$$

then plane angles:

$$BAC = p,\ BAD = q,\ CAD = r.$$

And from these the volume of the pyramid will be

$$\frac{1}{6}abd\sqrt{1 - \cos^2 p - \cos^2 q - \cos^2 r + 2\cos p \cos q \cos r},$$

which is reduced to the following form:

$$\frac{1}{3}abd\sqrt{\sin\frac{p+q+r}{2}\sin\frac{p+q-r}{2}\sin\frac{p+r-q}{2}\sin\frac{q+r-p}{2}};$$

whence it is clear that in order for the result to be a real area two of the three plane angles, p, q and r, which come together in any solid angle, when taken together must be greater than the third.