XXIII.

Continuatio Fragmentorum ex Adversariis mathematicis depromtorum.

(Conf. supra pagg. 157 ad 266.)

I. Supplementa numerorum doctrinae.

91.

(Lexell.)

PROBLEMA. Invenire numeros p, q, r, s, ut haec formula

$$\frac{\lambda(pp+ss)(qq+rr)}{pqrs(pp-ss)(qq-rr)}$$

fiat quadratum.

SOLUTIO. I. Primo ponatur

$$pp+ss=(aa+bb)(xx+yy) \quad \text{et}$$
$$qq+rr=(cc+dd)(xx+yy)$$

eritque

$$p=ax+by \quad \text{et} \quad q=cx+dy$$
$$s=bx-ay \quad\quad r=dx-cy;$$

quo facto, quadratum esse debet haec formula:

$$\frac{\lambda(aa+bb)(cc+dd)}{pqrs(p+s)(p-s)(q+r)(q-r)}.$$

II. Ut numerus factorum diminuatur, statuatur $r=s$, sive

$$dx-cy=bx-ay, \quad \text{unde fit } \frac{x}{y}=\frac{a-c}{b-d};$$

fiat ergo

$$x=a-c \quad \text{et} \quad y=b-d,$$

unde colligitur

$$p=aa-ac+bb-bd, \quad q=ac-cc+bd-dd \quad \text{et}$$
$$s=r=ab-bc-ab+ad=ad-bc, \text{ hincque}$$
$$p+s=aa-ac+ad+bb-bc-bd, \quad q+r=ac-bc-cc+bd+ad-dd$$
$$p-s=aa-ac-ad+bb+bc-bd, \quad q-r=ac+bc-cc+bd-ad-dd.$$

Formula ergo quadratum reddenda erit

$$\frac{\lambda(aa+bb)(cc+dd)}{pq(p+s)(p-s)(q+r)(q-r)}.$$

III. Fiat porro $p=cc+dd$, sive $cc+dd=aa-ac+bb-bd$, ad quam resolvendam statuatur $d=a$, eritque $cc=-ac+bb-ba$, sive $0=-cc-ac+bb-ab$, seu $cc+ac-bb+ab=0$, quae per $c+b$ divisa dat $a+c-b=0$, unde fit $c=b-a$ existente $d=a$. Habebimus ergo

$$p=cc+dd, \quad q=(3a-b)(b-a), \quad r=s=aa+ab-bb,$$

unde fit $p+s=a(3a-b)$, $p-s=(b-a)(2b-a)$, $q+r=(2a-b)(2b-a)$, $q-r=a(3b-4a)$.

Consequenter formula quadratum reddenda erit

$$\frac{\lambda(aa+bb)}{(3a-b)(b-a)a(3a-b)(b-a)(2b-a)(2a-b)(2b-a)a(3b-4a)},$$

quae reducitur ad hanc formam $\frac{\lambda(aa+bb)}{(2a-b)(3b-4a)}=\square$, ita ut habeatur haec conditio

$$\lambda(2a-b)(3b-4a)(aa+bb)=\square.$$

Nota. Si N° III posuissemus $d=-a$, habuissemus $cc-bb+ac-ab=0$, quae per $c-b$ divisa praebet $c+b+a=0$, sive $c=-a-b$, unde porro fit

$$p=(a+b)^2+aa=cc+dd,\quad q=-(3a+b)(a+b),\quad r=s=-(aa-ab-bb)$$

$$p+s=(2b+a)(b+a),\quad p-s=a(3a+b),\quad q+r=-a(4a+b),\quad q-r=-(2a+b)(2b+a)$$

Unde quadratum esse debet haec forma

$$\frac{\lambda(aa+bb)}{-(3a+b)(a+b)(2b+a)(b+a)a(3a+b)a(4a+3b)(2a+b)(2b+a)};$$

sicque quaestio reducitur ad hanc formam $-\lambda(aa+bb)(2a+b)(4a+3b)=\square$. Hic imprimis notatu dignum occurrit, quod per positionem tertiam, qua fecimus $p=cc+dd$, praeter expectationem, quatuor paria simplicium factorum ex calculo discesserunt.

Conditioni tertiae $p=cc+dd$ sequenti modo generaliter satisfieri potest: Quum sit

$cc+dd=aa-ac+bb-bd$, erit $cc+ac-aa=bb-bd-dd$, sive

$$(2c+a)^2-5aa=(2b-d)^2-5dd,\quad \text{sive}\quad (2c+a)^2-(2b-d)^2=5(aa-dd)\quad \text{et}$$

$$(2c+a-2b+d)(2c+a+2b-d)=5(a+d)(a-d)=5mntu;$$

unde colligitur $2c+a-2b+d=nu$, $2c+a+2b-d=5mt$ et $a+d=mu$ et $a-d=nt$. Ex his concluditur

$$a=\frac{mu+nt}{2}\quad \text{et}\quad d=\frac{mu-nt}{2};$$

inde vero $4c+2a=5mt+nu$ et $4b-2d=5mt-nu$, unde fit

$$c=\frac{(5m-n)t+(n-m)u}{4}\quad \text{et}\quad b=\frac{(5m-n)t-(n-m)u}{4}.$$

Hinc $p=[5(5mm-2mn+nn)tt-2(5mm-2mn+nn)tu+(5mm-2mn+nn)uu]:16$

$q=[-5(5mm-2mn+nn)tt+6(5mm-2mn+nn)tu-(5mm-2mn+nn)uu]:16$

sive $$p=\frac{5mm-2mn+nn}{16}(5tt-2tu+uu)$$

$$q=-\frac{(5mm-2mn+nn)}{16}(5tt-6tu+uu)=-\frac{(5mm-2mn+nn)}{16}(t-u)(5t-u)$$

$$r=s=\frac{5mm-2mn+nn}{16}(-5tt+uu).$$

Sit brevitatis gratia $\frac{(5mm-2mn+nn)}{16}=C$, ut sit

$$p=C(5tt-2tu+uu),\quad q=-C(t-u)(5t-u),\quad r=s=C(-5tt+uu)$$

eritque
$$p+s=-2Cu(t-u),\quad p-s=2Ct(5t-u)$$
$$q+r=-2Cu(3t-u),\quad q-r=2Ct(5t-3u)$$
$$aa+bb=C(5tt+2tu+uu);$$

quare formula quadratum reddenda est

$$\frac{\lambda(5tt+2tu+uu)}{(t-u)(5t-u)\,.\,-2u(t-u)2t(5t-u)\,.\,-2u(3t-u)2t(5t-3u)},$$

quae reducitur ad hanc conditionem: $\lambda(5tt+2tu+uu)(3t-u)(5t-3u)=\square$. Statuatur $u=\nu-t$, fietque $\lambda(4tt+\nu\nu)(4t-\nu)(8t-3\nu)=\square$; seu posito $2t=w$ erit $\lambda(ww+\nu\nu)(2w-\nu)(4w-3\nu)=\square$; quo facto habebitur $p=ww+(w-\nu)^2$, $q=(w-\nu)(3w-\nu)$ et $r=s=\nu\nu-\nu w-ww$.
Quae solutio cum praecedente prorsus congruit, ex quo patet illam solutionem multo esse generaliorem, quam initio videbatur.

Hinc alius modus solvendi colligitur: Ponatur $p+s=\alpha\beta$, $p-s=\varepsilon\zeta$, $q+r=\alpha\gamma$, $q-r=\varepsilon\eta$; tum vero $q=\beta\zeta$. Hinc ob $r=s$ fit $\frac{\alpha}{\varepsilon}=\frac{\zeta-\eta}{\beta-\gamma}$; ideoque sumatur $\alpha=\eta-\zeta$ et $\varepsilon=\gamma-\beta$; deinde $2\beta\zeta=\alpha\gamma+\varepsilon\eta$, habebitur $2\beta\zeta=2\eta\gamma-\gamma\zeta-\beta\eta$ et $\frac{\beta}{\gamma}=\frac{2\eta-\zeta}{2\zeta+\eta}$. Statuatur ergo $\beta=\frac{2\eta-\zeta}{5}$, $\gamma=\frac{2\zeta+\eta}{5}$, $\alpha=\eta-\zeta$, $\varepsilon=\frac{3\zeta-\eta}{5}$,

ergo
$$p=\frac{7\zeta\zeta-2\zeta\eta+\eta\eta}{5},\quad q=\frac{\zeta(2\eta-\zeta)}{5},\quad r=s=\frac{\eta\eta-\eta\zeta-\zeta\zeta}{5},$$

consequenter
$$pp+ss=\frac{(5\zeta\zeta-6\zeta\eta+2\eta\eta)(\zeta\zeta+\eta\eta)}{25}\quad \text{et}$$
$$qq+rr=\frac{(2\zeta\zeta-2\zeta\eta+\eta\eta)(\zeta\zeta+\eta\eta)}{25},$$

unde praecedens solutio nascitur. Imprimis hic notetur, totum negotium pendere ab his tribus rationibus: $\alpha:\varepsilon$, $\beta:\gamma$, et $\zeta:\eta$, neque ipsas quantitates absolutas in computum venire.

Solutio generalior.

Maneat
$$p+s=\alpha\beta,\quad p-s=\varepsilon\zeta,\quad q+r=\alpha\gamma,\quad q-r=\varepsilon\eta,\quad \text{ut sit}$$
$$p=\frac{\alpha\beta+\varepsilon\zeta}{2},\quad s=\frac{\alpha\beta-\varepsilon\zeta}{2};\quad q=\frac{\alpha\gamma+\varepsilon\eta}{2},\quad r=\frac{\alpha\gamma-\varepsilon\eta}{2};$$

at sit $r:s=f:g$ et $q=h\beta\zeta$; erit primo
$$\alpha\gamma-\varepsilon\eta:\alpha\beta-\varepsilon\zeta=f:g;\quad f\alpha\beta-f\varepsilon\zeta=g\alpha\gamma-g\varepsilon\eta\quad \text{et}\quad \alpha(f\beta-g\gamma)=\varepsilon(f\zeta-g\eta).$$

Ponatur ergo $\alpha=f\zeta-g\eta$ et $\varepsilon=f\beta-g\gamma$; deinde habemus
$$2h\beta\zeta=\alpha\gamma+\varepsilon\eta=f\gamma\zeta-2g\gamma\eta+f\beta\eta,\quad \text{unde}\quad \beta(2h\zeta-f\eta)=\gamma(f\zeta-2g\eta).$$

Ponatur ergo $\beta=f\zeta-2g\eta$ et $\gamma=2h\zeta-f\eta$ eritque
$$\alpha=f\zeta-g\eta\quad \text{et}\quad \varepsilon=(ff-2gh)\zeta-fg\eta.$$

Hinc ergo consequimur
$$p+s=ff\zeta\zeta-3fg\zeta\eta+2gg\eta\eta,\qquad p-s=(ff-2gh)\zeta\zeta-fg\zeta\eta$$

atque
$$q+r=2fh\zeta\zeta-(ff+2gh)\zeta\eta+fg\eta\eta,\quad q-r=(ff-2gh)\zeta\eta-fg\eta\eta;$$

unde fit
$$p=(ff-gh)\zeta\zeta-2fg\zeta\eta+gg\eta\eta,\qquad s=g(h\zeta\zeta-f\zeta\eta+g\eta\eta)$$
$$q=h\zeta(f\zeta-2g\eta)\qquad r=f(h\zeta\zeta-f\zeta\eta+g\eta\eta)$$

$$pp+ss=(f^4-2ffgh+2gghh)\zeta^4-2fg(2ff-gh)\zeta^3\eta+7ffgg\zeta\zeta\eta\eta-6fg^3\zeta\eta^3+2g^4\eta^4$$
$$qq+rr=2ffhh\zeta^4-2fh(ff+2gh)\zeta^3\eta+(f^4+2ffgh+4gghh)\zeta\zeta\eta\eta-2f^3g\zeta\eta^3+ffgg\eta^4,$$

quae forma ut divisibilis fiat per $p=(ff-gh)\zeta\zeta-2fg\zeta\eta+gg\eta\eta$, hae duae conditiones requiruntur:
$$\text{Primo}\ ff+gh=0,\quad \text{secundo}\ -3f^4+ffgh+4gghh=0;$$

tum vero quotus erit
$$ff\eta\eta+\left(\frac{3ffh}{g}+4hh\right)\zeta\zeta.$$

At prior conditio dat $gh=-ff$, vel et altera conditio, quae est
$$(ff+gh)(4gh-3ff)=0,$$

eo ipso impletur. Ita sit habeamus $h=-\frac{ff}{g}$; tum vero quotus erit
$$ff\eta\eta+hh\zeta\zeta.$$

Deinde vero ob $gh=-ff$, formula $pp+ss$ fit

$$=5f^4\zeta^4-6f^3g\zeta^3\eta+7ffgg\zeta\zeta\eta\eta-6fg^3\zeta\eta^3+2g^4\eta^4,$$

cujus factores sunt $(ff\zeta\zeta+gg\eta\eta)(5ff\zeta\zeta-6fg\zeta\eta+2gg\eta\eta)$. Quibus valoribus substitutis formula nostra quadratum reddenda fiet

$$\frac{\lambda(5ff\zeta\zeta-6fg\zeta\eta+2gg\eta\eta)(ff\zeta\zeta+gg\eta\eta)(ff\eta\eta+hh\zeta\zeta)}{hfg\eta(2h\zeta-f\eta)},$$

quae ut solubilis fiat, necesse est, ut bini superiores factores

$$ff\zeta\zeta+gg\eta\eta \quad \text{et} \quad ff\eta\eta+hh\zeta\zeta$$

coalescant, quod fit ponendo $ff:hh=gg:ff$, quod sponte evenit ob $ff=-gh$, ita ut res huc redeat

$$\frac{\lambda(5ff\zeta\zeta-6fg\zeta\eta+2gg\eta\eta)}{f\eta(f\eta-2h\zeta)} \quad \text{vel} \quad \frac{\lambda(5ff\zeta\zeta-6fg\zeta\eta+2gg\eta\eta)}{g\eta(2f\zeta+g\eta)},$$

quae iterum a praecedente non discrepat, nisi quod hic sit $f\zeta$ et $g\eta$ quod supra erat ζ et η.

A. m. T. I. p. 17 – 21.

92.

(J. A. Euler.)

Theorema. Si formula $aapp+b\beta qq$ ducatur in formulam $abrr+\alpha\beta ss$, productum erit

$$aa\alpha\beta pprr+a\alpha\alpha\beta ppss+abb\beta qqrr+\alpha b\beta\beta qqss=\alpha b(aapprr+\beta\beta qqss)+a\beta(\alpha\alpha ppss+bbqqrr)=$$
$$\alpha b(apr\pm\beta qs)^2+a\beta(\alpha ps\mp bqr)^2.$$

Hujus ergo producti forma est $\alpha bxx+a\beta yy$ existente

$$x=apr\pm\beta qs \quad \text{et} \quad y=\alpha ps\mp bqr.$$

A. m. T. I. p. 130.

93.

(Lexell.)

Problema. Si fuerit $x^3=a$ et proponatur formula $fxx+gx+h$, quaerere multiplicatorem $pxx+qx+r$, ut productum fiat numerus rationalis.

Solutio. Cum productum sit

$$fpx^4+(fq+gp)x^3+(fr+hp+gq)xx+(gr+hq)x+hr$$

ob $x^3=a$, hoc productum reducitur ad sequentem formam

$$(fr+hp+gq)xx+(fpa+gr+hq)x+fq+gp+hr=0$$

unde
$$r=\frac{-hp-gq}{f}; \quad r=\frac{-fpa-hq}{g}; \quad r=\frac{-fq-gp}{h}$$

atque
$$hpg+g^2q=f^2pa+fhq; \quad p(hg-f^2)=q(fh-g^2); \quad \frac{p}{q}=\frac{fh-gg}{hg-ff};$$

hinc $p=fh-gg$, $q=hg-ff$, $r=gf-hh$, atque productum quaesitum erit $=3fgh-f^3-g^3-h^3$.

Hujus ope radices cubicas numerorum licebit ad fractiones continuas revocare. *Exempl.* Proponatur $x^3=2$, ut sit $x=\sqrt[3]{2}$; notetur esse proxime $x=1\frac{1}{4}$ et $xx=1\frac{1}{2}$, et nunc more consueto fractio $\frac{x}{1}$ in fractionem continuam convertatur ....

A. m. T. I. p. 176.

94.

(N. Fuss.)

Si fuerit $x = z^4 - 6zz + 1$ et $y = 4z^3 - 4z$, erit $xx + yy = (zz + 1)^4$. At vero

$$x + y = z^4 + 4z^3 - 6zz - 4z + 1,$$

quae formula resolvitur in hos factores

$$[zz + (2 + 2\sqrt{2})z - 1]\,[zz + (2 - 2\sqrt{2})z - 1]$$

quod si jam $x + y$ debeat esse quadratum, fiat uterque factor quadratum ponendo

$$zz + (2 + 2\sqrt{2})z - 1 = (z + p + q\sqrt{2})^2$$
$$zz + (2 - 2\sqrt{2})z - 1 = (z + p - q\sqrt{2})^2$$

tum enim erit $x + y = (zz + 2pz + pp - 2qq)^2$. Jam evolvatur alterutra harum positionum, et termini rationales inter se seorsim aequantur et irrationales:

$$2z - 1 = 2pz + pp + 2qq$$
$$2z\sqrt{2} = 2qz\sqrt{2} + 2pq\sqrt{2}.$$

Prior aequatio dat $2z - 2pz = 1 + pp + 2qq$, unde $z = \frac{1 + pp + 2qq}{2(1 - p)}$; ex altera autem aequatione per $2\sqrt{2}$ divisa fit $z = qz + pq$, hincque $z = \frac{pq}{1 - q}$; qui duo valores inter se aequati praebent $p = \frac{q \pm \sqrt{(2q^4 - 1)}}{1 + q}$.

Si hic capiatur $q = 13$, fiet $p =$ vel 18, vel $= \frac{-113}{7}$. Poni etiam posset $q = -13$, fieretque

$$p = \frac{-13 \pm 239}{-12};$$

hinc vel $p = 21$, vel $= \frac{113}{6}$. Si sumatur $q = 13$ et $p = \frac{-113}{7}$, reperietur $z = \frac{1469}{84}$.

Haec methodus ad sequentem redire videtur, quae resolutione in factores non indiget et ita se habet. Sit formula proposita quadratum efficienda in genere $z^4 + az^3 + bzz + cz + d$, cujus radix ponatur $zz + pz + r$, ita ut fieri debeat

$$\begin{array}{l} z^4 + 2pz^3 + 2rzz + 2prz + rr \\ \qquad\qquad\quad + ppzz \\ \hline -z^4 - az^3 - bzz - cz - d = 0 \end{array}$$

ubi cum primi termini se destruant, termini secundi et tertii ad nihilum redigantur, unde per zz dividendo fiet $(2p - a)z + 2r + pp - b = 0$, ideoque $z = \frac{2r + pp - b}{a - 2p}$. Simili vero modo termini quarti et quinti conjunctim tollantur, unde fiet $(2pr - c)z + rr - d = 0$, indeque $z = \frac{rr - d}{c - 2pr}$. Hi duo valores ipsius z inter se aequati dabunt

$$r = c + bp - p^3 \pm \sqrt{(cc + a(ad - bc) + bbpp + acpp - 4dpp - 2bp^4 + p^6)}.$$

Nostro autem casu erat $a = 4$, $b = -6$, $c = -4$, $d = 1$; hinc formula radicalis evadit

$$\sqrt{(-64 + 16pp + 12p^4 + p^6)} \quad \text{sive} \quad \sqrt{(pp + 4)(p^4 + 8pp - 16)},$$

quae autem formula nullo modo tractari potest, unde patet priorem methodum non reduci ad hanc posteriorem, ideoque eo magis attentionem merere.

A. m. T. I. p. 276. 277.

95.

(N. Fuss.)

Methodus facilis hujusmodi quaestiones solvendi: Quaerantur numeri x et y tales, ut formula mx^4+ny^4 divisibilis fiat per datum numerum N.

Primum observandum, hoc fieri non posse, nisi fuerit vel $N=maa+nbb$, vel $N=aa+mnbb$. Pro casu priore quaeratur quadratum $kk=\lambda N\pm ab$, quod si fieri nequeat, quaestio est impossibilis. Sin autem k inventum fuerit, erit

$$x=\alpha N\pm ap,\quad \text{vel etiam}\quad x=\alpha N\pm kq$$
$$y=\beta N\pm kp,\quad\quad y=\beta N\pm bq$$

ubi α, β, p, q pro lubitu sumuntur.

Pro altero casu quaeratur quadratum $kk=\lambda mN\pm mab$, tum vero erit ut ante $x=\alpha N\pm ap$ et $y=\beta N\pm kp$, vel etiam $x=\alpha N\pm kq$, $y=\beta N\pm bq$. Sit $m=2$ et $n=1$, ut formula $2x^4+y^4$ divisibilis fiat per $N=2aa+bb$. Sumatur $a=4$ et $b=1$, erit $N=33$. Quaeratur ergo $kk=33\lambda\pm 4$, quod fit si $\lambda=0$, eritque $k=2$; erit ergo $x=33\alpha\pm 4p$ et $y=33\beta\pm 2p$.

A. m. T. II. p. 148. 149.

96.

(N. Fuss.)

Definitio. Proposito numero quocunque integro a, denotet πa multitudinem numerorum ipso a minorum ad eumque primorum; ita erit $\pi 1=1$, $\pi 2=1$, $\pi 3=2$, $\pi 4=2$, $\pi 5=4$, $\pi 6=2$ etc. Unde patet, si a fuerit numerus primus, fore $\pi a=a-1$. Quo magis autem numerus a fuerit compositus, eo minor erit πa. Quemadmodum autem pro quovis numero a inveniri queat valor πa, regulam quidem olim dedi, ejus vero demonstrationem multo simpliciorem hic sum traditurus.

Lemma. Proposito quocunque numero a, si formetur progressio arithmetica totidem terminorum, cujus differentia ad eum sit prima, ejusque singuli termini per a dividantur, omnia residua inter se erunt diversa, in iisque ergo occurrent omnes numeri ipso a minores, scil. $0, 1, 2, 3, 4 \ldots a-1$.

Demonstratio. Sit p primus terminus et q differentia ad a prima, erit progressio arithmetica

$$p,\ p+q,\ p+2q,\ \ldots\ p+(a-1)q.$$

Quod si jam singuli termini per a dividantur, facile patet omnia residua inde orta inaequalia esse debere. Si enim hi termini $p+\mu q$ et $p+\nu q$, ubi μ et ν minores sunt quam a, idem praeberent residuum, eorum differentia, quae est $(\mu-\nu)q$, foret per a divisibilis. At quia q est numerus primus ad a, deberet $\mu-\nu$, hoc est numerus ipso a minor, per eum esse divisibilis. Cum igitur omnia residua sint diversa, eorumque numerus $=a$, in iis necessario reperientur omnes numeri $0, 1, 2, 3$ etc. $\ldots a-1$; semper igitur unus horum numerorum per a erit divisibilis.

Praeparatio ad demonstrationem. Sint $1, \alpha, \beta, \gamma$ omnes numeri ipso a minores ad eumque primi, quorum ergo numerus per hypothesin $=\pi a$, inter quos ergo primus erit 1, et ultimus $a-1$. Hinc constituantur sequentes series:

1	α	β	$\gamma \ldots a-1$	a
$a+1$	$a+\alpha$	$a+\beta$	$a+\gamma \ldots a-1$	$2a$
$2a+1$	$2a+\alpha$	$2a+\beta$	$2a+\gamma \ldots 3a-1$	$3a$
$3a+1$	$3a+\alpha$	$3a+\beta$	$3a+\gamma \ldots 4a-1$	$4a$
. . .	. . .	. . .	. . .	. . .
$(n-1)a+1$,	$(n-1)a+\alpha$,	$(n-1)a+\beta$,	$(n-1)a+\gamma \ldots na-1$	na

Quemadmodum igitur hic prima series horizontalis continet omnes numeros ad a primos ab 0 usque ad a, ita secunda series continet omnes ad a primos usque ad $2a$, tertia vero omnes numeros ad a primos ab $2a$ usque ad $3a$, hocque modo hae series continuentur usque ad ultimam $(n-1)a+1$. Omnes igitur, conjunctim praebent omnes numeros ad a primos ab 0 usque ad na, quorum ergo numerus est $n\pi a$. Singulae autem series verticales erunt arithmeticae progressiones differentia a crescentes. His praemissis sequentia problemata facillime solventur.

Problema. Proposito numero quocunque a, investigare valores formularum πa^2, πa^3, πa^4, et in genere πa^m.

Solutio. In schemate superiore sumamus $n=a$, ut quaeratur πa^2, atque manifestum est omnes terminos illarum serierum, quia sunt primi ad a, etiam primos fore ad aa. Quare cum earum serierum numerus sit $n=a$, et cujusque terminorum numerus $=\pi a$, omnino habebimus $a.\pi a$, cui ergo aequalis πaa, ita ut sit $\pi aa = a\pi a$. Deinde sumto $n=aa$, ut sit $na=a^3$, quia iterum omnes termini sunt primi ad a^3, eorum numerus erit $aa\pi a$, ideoque $\pi a^3 = aa\pi a$. Atque in genere si sumatur $n=a^{m-1}$, ut fiat $na=a^m$, multitudo omnium numerorum ad a^m primorum erit

$$a^{m-1}\pi a.$$

Coroll. Si igitur a numerus primus, ideoque $\pi a = a-1$, erit

$$\pi a^2 = a(a-1), \quad \pi a^3 = aa(a-1) \quad \text{et} \ldots \pi a^m = a^{m-1}(a-1).$$

Problema. Propositis duobus numeris a et b inter se primis, pro quibus habeantur formulae πa et πb, invenire multitudinem omnium numerorum ad productum ab primorum ipsoque minorum, sive investigare valorem πab.

Solutio. In schemate superiore sumatur $n=b$, ut fiat $na=ab$; et quia series horizontales continent omnes numeros ad a primos ab 1 usque ad ab, quorum ergo numerus est $b\pi a$, jam consideretur prima series verticalis, quae est $1, a+1, 2a+1, \ldots (b-1)a+1$, quae quia est arithmetica, ejusque differentia a est prima ad b, numerus terminorum ad b primorum $=\pi b$. Hoc idem valet de reliquis seriebus verticalibus, quarum quaelibet πb continet terminos ad b primos. Quamobrem numerus omnium terminorum simul ad a et b primorum, ob numerum verticalium $=\pi a$, erit $=\pi a.\pi b$, ita ut sit $\pi ab = \pi a.\pi b$.

Hinc jam tabula pro omnibus numeris condi poterit:

$\pi 1 = 1$	$\pi 5 = 4$	$\pi 9 = 6$	$\pi 13 = 12$
$\pi 2 = 1$	$\pi 6 = 2$	$\pi 10 = 4$	$\pi 14 = 6$
$\pi 3 = 2$	$\pi 7 = 6$	$\pi 11 = 10$	$\pi 15 = 8$
$\pi 4 = 2$	$\pi 8 = 4$	$\pi 12 = 4$	$\pi 16 = 7$ etc.

Hinc porro patet, si fuerint a, b, c, d numeri inter se primi, tum fore $\pi abcd = \pi a.\pi b.\pi c.\pi d$. Hinc similiter, si proponatur numerus $a^\alpha b^\beta c^\gamma d^\delta = N$, erit $\pi N = a^{\alpha-1}\pi a.b^{\beta-1}\pi b.c^{\gamma-1}\pi c.d^{\delta-1}\pi d$.

A. m. T. III. p. 182 — 184.

II. Geometria.

97.

(Golovin.)

Problema. Invenire duas superficies, quarum alteram in alteram transformare liceat, ita ut in utraque singula puncta homologa easdem inter se teneant distantias.

* Solutio. Pro priori superficie sit (Fig. 61.) Z punctum ejus quodcunque determinatum per tres coordinatas

$$AT = t, \; TU = u, \; UZ = v.$$

In altera vero superficie idem punctum Z determinatum sit per ternas coordinatas $CX = x$, $XY = y$, $YZ = z$. Et quia per naturam superficierum quaelibet coordinata debet esse functio binarum variabilium, sint r et s hae duae variabiles a se invicem non pendentes, harumque functiones sint nostrae coordinatae. Nunc considerentur in utraque superficie duo puncta r, s, ipsi Z proxima, quorum illud r prodeat ex variatione solius r, alterum vero s oriatur ex variatione sola ipsius s, ac per conditionem problematis terna intervalla infinite parva Zr, Zs, rs utrinque debent esse aequalia. Pro puncto autem r in prima figura ternae coordinatae erunt

$$t + dr\left(\frac{dt}{dr}\right), \quad u + dr\left(\frac{du}{dr}\right), \quad v + dr\left(\frac{dv}{dr}\right).$$

Simili modo pro puncto s in prima figura ternae coordinatae erunt

$$t + ds\left(\frac{dt}{ds}\right), \quad u + ds\left(\frac{du}{ds}\right), \quad v + ds\left(\frac{dv}{ds}\right).$$

Hinc quadrata memoratorum intervallorum colliguntur

$$Zr^2 = dr^2\left(\left(\frac{dt}{dr}\right)^2 + \left(\frac{du}{dr}\right)^2 + \left(\frac{dv}{dr}\right)^2\right)$$

$$Zs^2 = ds^2\left(\left(\frac{dt}{ds}\right)^2 + \left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right)$$

$$rs^2 = \left(dr\left(\frac{dt}{dr}\right) - ds\left(\frac{dt}{ds}\right)\right)^2 + \left(dr\left(\frac{du}{dr}\right) - ds\left(\frac{du}{ds}\right)\right)^2 + \left(dr\left(\frac{dv}{dr}\right) - ds\left(\frac{dv}{ds}\right)\right)^2,$$

quod postremum quadratum reducitur ad hanc formam

$$rs^2 = Zr^2 + Zs^2 - 2dr\,ds\left(\left(\frac{dt}{dr}\right)\left(\frac{dt}{ds}\right) + \left(\frac{du}{dr}\right)\left(\frac{du}{ds}\right) + \left(\frac{dv}{dr}\right)\left(\frac{dv}{ds}\right)\right).$$

Quodsi jam loco t, u, v scribentur litterae x, y, z, habebuntur eadem intervalla pro altera figura, quae cum utrinque inter se debeant esse aequalia, habebimus has tres aequationes

$$\text{I.} \quad \left(\frac{dt}{dr}\right)^2 + \left(\frac{du}{dr}\right)^2 + \left(\frac{dv}{dr}\right)^2 = \left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2$$

$$\text{II.} \quad \left(\frac{dt}{ds}\right)^2 + \left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$$

$$\text{III.} \quad \left(\frac{dt}{dr}\right)\left(\frac{dt}{ds}\right) + \left(\frac{du}{dr}\right)\left(\frac{du}{ds}\right) + \left(\frac{dv}{dr}\right)\left(\frac{dv}{ds}\right) = \left(\frac{dx}{dr}\right)\left(\frac{dx}{ds}\right) + \left(\frac{dy}{dr}\right)\left(\frac{dy}{ds}\right) + \left(\frac{dz}{dr}\right)\left(\frac{dz}{ds}\right),$$

in quibus tribus aequationibus continetur solutio nostri problematis. Quemadmodum autem per methodos cognitas iis satisfieri oporteat, neutiquam patet, opusque maxime arduum videtur.

Huc autem superior analysis sequenti modo traduci poterit: Sint litterae J, G, H, item L, M, N functiones prioris tantum variabilis r, et statuantur nostrae coordinatae

$$\text{priores} \quad t = \int Jdr + Js \qquad \text{posteriores} \quad x = \int Ldr + Ls$$

$$u = \int Gdr + Gs \qquad y = \int Mdr + Ms$$

$$v = \int Hdr + Hs \qquad z = \int Ndr + Ns$$

unde differentialia eliciuntur $\left(\frac{dt}{dr}\right)=J+\frac{sdJ}{dr}$ et $\left(\frac{dt}{ds}\right)=J$, sicque de reliquis. Unde tres aequationes, quibus satisfieri oportet, erunt

$$\text{I.}\quad \left(J+\frac{sdJ}{dr}\right)^2+\left(G+\frac{sdG}{dr}\right)^2+\left(H+\frac{sdH}{dr}\right)^2=\left(L+\frac{sdL}{dr}\right)^2+\left(M+\frac{sdM}{dr}\right)^2+\left(N+\frac{sdN}{dr}\right)^2$$

$$\text{II.}\quad J^2+G^2+H^2=L^2+M^2+N^2$$

$$\text{III.}\quad J\left(J+\frac{sdJ}{dr}\right)+G\left(G+\frac{sdG}{dr}\right)+H\left(H+\frac{sdH}{dr}\right)=L\left(L+\frac{sdL}{dr}\right)+M\left(M+\frac{sdM}{dr}\right)+N\left(N+\frac{sdN}{dr}\right)$$

quae manifesto ad tres sequentes aequalitates reducuntur

$$\text{I.}\quad J^2+G^2+H^2=L^2+M^2+N^2$$

$$\text{II.}\quad JdJ+GdG+HdH=LdL+MdM+NdN$$

$$\text{III.}\quad dJ^2+dG^2+dH^2=dL^2+dM^2+dN^2$$

quarum secunda jam in prima continetur; ita ut tantum duae conditiones adimplendae supersint.

Quo hae formulae magis evolvantur, statuamus $J^2+G^2+H^2=pp$, erit quoque $L^2+M^2+N^2=pp$. Quocirca ponamus

$$J=p\sin m\sin n,\qquad G=p\cos m\sin n,\qquad H=p\cos n$$
$$L=p\sin\mu\sin\nu,\qquad M=p\cos\mu\sin\nu,\qquad N=p\cos\nu.$$

Hocque modo alteri conditioni jam erit satisfactum. Pro altera autem habebimus:

$$(dp\sin m\sin n+pdm\cos m\sin n+pdn\sin m\cos n)^2+(dp\cos m\sin n-pdm\sin m\sin n+pdn\cos m\cos n)^2+(dp\cos n-pdn\sin n)^2=$$
$$(dp\sin\mu\sin\nu+pd\mu\cos\mu\sin\nu+pd\nu\sin\mu\cos\nu)^2+(dp\cos\mu\sin\nu-pd\mu\sin\mu\sin\nu+pd\nu\cos\mu\cos\nu)^2+(dp\cos\nu-pd\nu\sin\nu)^2$$

quae reducitur ad sequentem formam multo simpliciorem

$$dp^2+p^2dm^2\sin^2n+p^2dn^2=dp^2+p^2d\mu^2\sin^2\nu+p^2d\nu^2$$

sive ad hanc $dm^2\sin^2n+dn^2=d\mu^2\sin^2\nu+d\nu^2$. Sumere igitur licet quatuor angulos m, n et μ, ν, utcunque a variabili r pendentes, dummodo sit

$$dm^2\sin^2n+dn^2=d\mu^2\sin^2\nu+d\nu^2$$

sive tribus m, n et ν pro arbitrio assumtis, quartus μ ita definiatur, ut sit $d\mu=\frac{\sqrt{(dm^2\sin^2n+dn^2-d\nu^2)}}{\sin\nu}$. Vel etiam introducto novo angulo θ functione ipsius r, tantum capi poterit $dm=\frac{\sqrt{(d\theta^2-dn^2)}}{\sin n}$ et $d\mu=\frac{\sqrt{(d\theta^2-d\nu^2)}}{\sin\nu}$. Quo facto ternae coordinatae pro utraque superficie quaesita erunt:

pro priori:
$$\begin{cases} t=\int pdr\sin m\sin n+ps\sin m\sin n;\\ u=\int pdr\cos m\sin n+ps\cos m\sin n;\\ v=\int pdr\cos n+ps\cos n,\end{cases}$$
pro posteriori:
$$\begin{cases} x=\int pdr\sin\mu\sin\nu+ps\sin\mu\sin\nu\\ y=\int pdr\cos\mu\sin\nu+ps\cos\mu\sin\nu\\ z=\int pdr\cos\nu+ps\cos\nu,\end{cases}$$

ubi denuo pro p functionem quamcunque ipsius r capere licet.

Adnotatio. Probe autem notari convenit hic alteram superficiem non pro data assumi licere, saltem non patet quomodo functiones p, m et n assumi debeant, ut prior superficies datam obtineat figuram v. g. sphaericam. Cum enim in utrisque formulis binae variabiles r et s in infinitum augeri queant, facile patet utramque superficiem necessario in infinitum protendi, neque hanc extensionem per quaepiam imaginaria tolli posse. Quamobrem figura sphaerica neque ulla alia figura in spatio finito subsistens in his formulis contenta esse potest. Quod autem ad figuras terminatas seu undique clausas attinet, judicium de iis aliter instituendum videtur. Statim enim atque figura solida undique est clausa, nullam amplius mutationem patitur; quemadmodum ex notis illis figuris corporeis, quae corpora regularia vocari solent, intelligere licet. Unde quatenus superficies sphaerica est integra, nullam mutationem admittit. Hinc patet, eatenus hujusmodi figuras mutari posse, quatenus non sunt integrae seu undique clausae. Interim patet hemisphaerii figuram certe esse mutabilem; cujusmodi autem mutationes recipere possit, problema videtur difficillimum.

A. m. T. I. p. 10—13.

98.

(Lexell.)

Problema. Ex datis aliquot applicatis aeque distantibus, aream interceptam curvae proxime definire.

(Fig. 62). I. Area $AaBb = AB\left(\frac{Aa + Bb}{2}\right)$

II. » $AaCc = AC\left(\frac{Aa + 4Bb + Cc}{6}\right)$

III. » $AaDd = AD\left(\frac{Aa + 3Bb + 3Cc + Dd}{8}\right)$

IV. » $AaEe = AE\left(\frac{7Aa + 32Bb + 12Cc + 32Dd + 7Ee}{90}\right)$

et ita porro. Sit

$$\text{area } AaXx = AX(\alpha Aa + \beta Bb + \gamma Cc + \delta Dd + \ldots + \xi Xx)$$

erit

$$\text{I.}\quad \alpha + \beta + \gamma + \delta + \text{ etc. } = 1$$

$$\text{II.}\quad \beta + 2\gamma + 3\delta + 4\varepsilon + \text{ etc. } = \frac{4}{2} = \frac{2^2}{2}$$

$$\text{III.}\quad \beta + 2^2\gamma + 3^2\delta + 4^2\varepsilon + \text{ etc. } = \frac{2^4}{3}$$

$$\text{IV.}\quad \beta + 2^3\gamma + 3^3\delta + 4^3\varepsilon + \text{ etc. } = \frac{2^6}{4}$$

$$\text{V.}\quad \beta + 2^4\gamma + 3^4\delta + 4^4\varepsilon + \text{ etc. } = \frac{2^8}{5}$$

et sic porro.

A. m. T. I. p. 128.

99.

(J. A. Euler.)

In logarithmica (Fig. 63), cujus subtangens $AD = 1$, ab applicata $AB = 1$ longitudo curvae in infinitum extensae BU superat axem AV etiam in infinitum productum quantitate $\sqrt{2} - 1 - l\frac{\sqrt{2}+1}{2}$. Nam sit abscissa $AP = x$ et $PM = y$, erit $y = e^{-x}$, hinc $dy = -e^{-x}dx$, et elementum arcus $= dx\sqrt{(1 + e^{-2x})}$, hinc

$$BM - AP = \int dx\left(\sqrt{(1 + e^{-2x})} - 1\right)$$

quod integrale ab $x = 0$ usque ad $x = \infty$ extendi debet.

Ponatur $\sqrt{(1 + e^{-2x})} - 1 = z$, fiet $e^{-2x} = 2z + zz$ et $-2x = l(2z + zz)$; ubi pro $x = 0$ habemus $z = \sqrt{2} - 1$, et pro $x = \infty$ fit $z = 0$; hinc differentiando

$$-dx = \frac{dz(1+z)}{2z + zz}, \text{ et formula nostra fit } -\int\frac{dz(1+z)}{2+z}$$

quae integrari debet ab $z = \sqrt{2} - 1$ usque ad $z = 0$, vel nostra formula erit

$$-dz\left(1 - \frac{1}{2+z}\right), \text{ ergo integrando } -z + l(2+z) + \sqrt{2} - 1 - l(1+\sqrt{2}).$$

Nunc fiat $z = 0$, et quantitas quaesita erit

$$\sqrt{2} - 1 + l2 - l(1+\sqrt{2}) = \sqrt{2} - 1 + l\frac{2}{\sqrt{2}+1}.$$

A. m. T. I. p. 258.

100.

(J. A. Euler.)

Problema. (Fig. 64.) Pro hyperbola, cujus semiaxis $AC = a$, posito $AP = x$, $PM = y$, sit $ny = \sqrt{(2ax + xx)}$, * et ex M ad asymtotam CN ducatur MN axi parallela, invenire excursum rectae CN supra curvam AM, quando punctum M in infinitum promovetur.

Posito $x = \infty$ fit $ny = x$, hinc $\operatorname{tang} ACN = \frac{1}{n}$ et $\sin ACN = \frac{1}{\sqrt{(1+nn)}} = \frac{PM}{CN} = \frac{y}{CN}$, ergo $CN = y\sqrt{(1+nn)}$. Tum vero habemus $nnyy + aa = (a+x)^2$, ergo

$$x = \sqrt{(nnyy + aa)} - a, \quad \text{unde} \quad dx = \frac{nnydy}{\sqrt{(nnyy+aa)}};$$

hinc arcus $AM = \int dy \sqrt{\left(1 + nn - \frac{nnaa}{nnyy+aa}\right)}$. Hinc

$$CN - AM = \int dy \left(\sqrt{(nn+1)} - \sqrt{\left(nn + 1 - \frac{nnaa}{nnyy+aa}\right)}\right).$$

Ponatur nunc $\nu = \sqrt{(nn+1)} - \sqrt{\left(nn + 1 - \frac{nnaa}{nnyy+aa}\right)}$, erit

$$\frac{-nnaa}{nnyy+aa} = -2\nu\sqrt{(nn+1)} + \nu\nu, \quad \text{sive}$$

$$\frac{1}{2\nu\sqrt{(nn+1)} - \nu\nu} = \frac{yy}{aa} + \frac{1}{nn}, \quad \text{ergo}$$

$$y = \frac{a}{n}\sqrt{\frac{nn - 2\nu\sqrt{(nn+1)} + \nu\nu}{2\nu\sqrt{(nn+1)} - \nu\nu}}.$$

Per logarithmos autem erit

$$2ly - 2la = l\left(nn - 2\nu\sqrt{(nn+1)} + \nu\nu\right) - 2ln - l\left(2\nu\sqrt{(nn+1)} - \nu\nu\right)$$

$$\frac{dy}{y} = \frac{-d\nu\sqrt{(1+nn)} + \nu d\nu}{nn - 2\nu\sqrt{(nn+1)} + \nu\nu} - \frac{d\nu\sqrt{(nn+1)} + \nu d\nu}{2\nu\sqrt{(nn+1)} - \nu\nu}$$

hinc autem vix quicquam concludi poterit.

Ineamus ergo aliam viam: Cum sit

$$CN - AM = \sqrt{(nn+1)}\int dy \left(1 - \sqrt{\left(1 - \frac{nnaa}{nn+1}\cdot\frac{1}{nnyy+aa}\right)}\right),$$

sit $\frac{nnaa}{nn+1}\cdot\frac{1}{nnyy+aa} = \cos^2\varphi$, erit $nnyy + aa = \frac{nnaa}{(nn+1)\cos^2\varphi}$, hinc

$$ny = \frac{a\sqrt{(nn\sin^2\varphi - \cos^2\varphi)}}{\cos\varphi\sqrt{(nn+1)}},$$

ubi casu $y = 0$ erit $\cos^2\varphi = \frac{nn}{nn+1}$ et $\cos\varphi = \frac{n}{\sqrt{(nn+1)}}$ et $\sin\varphi = \frac{1}{\sqrt{(nn+1)}}$, $\operatorname{tang}\varphi = \frac{1}{n}$, hinc $\varphi = ACN$, et pro $y = \infty$ erit $\varphi = 90^0$. Ergo integrari debet a $\varphi = ACN$, vel $\operatorname{tang}\varphi = \frac{1}{n}$, usque ad $\varphi = 90^0$, vel $\operatorname{tang}\varphi = \infty$. Est autem $ny = \frac{a\sqrt{(nn\operatorname{tang}^2\varphi - 1)}}{\sqrt{(nn+1)}}$. Ponatur $\operatorname{tang}\varphi = t$ et integrandum a $t = \frac{1}{n}$ usque ad $t = \infty$; at $\sin\varphi = \frac{t}{\sqrt{(1+tt)}}$. Hinc $CN - AM = \sqrt{(1+nn)}.\ \frac{ann}{n\sqrt{(nn+1)}}\int\frac{tdt}{\sqrt{(nntt-1)}}\left(1 - \frac{t}{\sqrt{(1+tt)}}\right)$, vel

$$CN - AM = \frac{a}{n}\sqrt{(nntt-1)} - \frac{a}{n}\int\frac{nnttdt}{\sqrt{(tt+1)(nntt-1)}}.$$

Est autem $$\frac{1}{\sqrt{(1+tt)}} = (1+tt)^{-\frac{1}{2}} = \frac{1}{t} - \frac{1}{2}\cdot\frac{1}{t^3} + \frac{1.3}{2.4}\cdot\frac{1}{t^5} - \frac{1.3.5}{2.4.6}\cdot\frac{1}{t^7} + \text{ etc.}$$

Erit $CN - AM = na\left(\frac{1}{2}\int\frac{dt}{t\sqrt{(nntt-1)}} - \frac{1.3}{2.4}\int\frac{dt}{t^3\sqrt{(nntt-1)}} + \frac{1.3.5}{2.4.6}\int\frac{dt}{t^5\sqrt{(nntt-1)}} - \text{ etc.}\right)$. Ubi notandum si scribatur $t=\frac{1}{u}$, fore $\int\frac{dt}{t\sqrt{(nntt-1)}}=\int\frac{-du}{\sqrt{(nn-uu)}}=Arc.\cos\frac{u}{n}=Arc.\cos\frac{1}{nt}$, et facto $t=\infty$ erit hoc integrale $=\frac{\pi}{2}$. Deinde

$$\int\frac{dt}{t^3\sqrt{(nntt-1)}}=\int\frac{-uudu}{\sqrt{(nn-uu)}},\quad \int\frac{dt}{t^5\sqrt{(nntt-1)}}=\int\frac{-u^4du}{\sqrt{(nn-uu)}},\quad \int\frac{dt}{t^7\sqrt{(nntt-1)}}=\int\frac{-u^6du}{\sqrt{(nn-uu)}}$$

etc. Fingatur $\int\frac{-u^{\lambda+2}du}{\sqrt{(nn-uu)}}=A\int\frac{-u^\lambda du}{\sqrt{(nn-uu)}}+Bu^{\lambda+1}\sqrt{(nn-uu)}$, ubi terminus algebraicus fit $=0$ tam si $u=n$ quam si $u=0$, ergo ob $A=\frac{(\lambda+1)nn}{\lambda+2}$ erit

$$\int\frac{-u^{\lambda+2}du}{\sqrt{(nn-uu)}}=\frac{(\lambda+1)nn}{\lambda+2}\int\frac{-u^\lambda du}{\sqrt{(nn-uu)}}.$$

Cum nunc esset $\int\frac{-du}{\sqrt{(nn-uu)}}=\frac{\pi}{2}$, erit

$$\int\frac{-uu\,du}{\sqrt{(nn-uu)}}=\frac{1}{2}\cdot\frac{\pi}{2}\cdot nn,\quad \int\frac{-u^4du}{\sqrt{(nn-uu)}}=\frac{1.3}{2.4}\cdot\frac{\pi}{2}\cdot n^4,\quad \int\frac{-u^6du}{\sqrt{(nn-uu)}}=\frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}\cdot n^6 \text{ etc.}$$

quamobrem habebimus

$$CN-AM=\frac{\pi}{2}\cdot na\left(\frac{1}{2}-\frac{1.3}{2.4}\cdot\frac{1}{2}\cdot nn+\frac{1.3.5}{2.4.6}\cdot\frac{1.3}{2.4}\cdot n^4-\frac{1.3.5.7}{2.4.6.8}\cdot\frac{1.3.5}{2.4.6}\cdot n^6+\text{ etc.}\right).$$

Unde excessus in problemate quaesitus $CN-AM$ pro infinito erit

$$\frac{\pi}{2}\cdot na\left(\frac{1}{2}-\frac{1^2}{2^2}\cdot\frac{3}{4}n^2+\frac{1^2.3^2}{2^2.4^2}\cdot\frac{5}{6}n^4-\frac{1^2.3^2.5^2}{2^2.4^2.6^2}\cdot\frac{7}{8}n^6+\text{ etc.}\right)$$

quae si n fuerit unitate minus, valde convergit.

Sequenti autem modo hoc problema elegantius solvetur. Cum sit

$$CN-AM=\sqrt{(1+nn)}\int dy\left(1-\sqrt{\left(1-\frac{nnaa}{1+nn}\cdot\frac{1}{nnyy+aa}\right)}\right),$$

ponatur $\frac{nn}{nn+1}=m$ et $\frac{aa}{nnyy+aa}=uu$, erit $\frac{nnyy+aa}{aa}=\frac{1}{uu}$, hincque $y=\frac{a}{n}\cdot\frac{\sqrt{(1-uu)}}{u}$ atque

$$dy=-\frac{a}{n}\cdot\frac{du}{uu\sqrt{(1-uu)}};$$

ubi pro $y=0$ habemus $u=1$, et pro $y=\infty$, $u=0$. Unde fit

$$CN-AM=\frac{-a}{\sqrt{m}}\int\frac{du\,(1-\sqrt{(1-muu)})}{uu\sqrt{(1-uu)}}$$

ubi $\int\frac{du}{uu\sqrt{(1-uu)}}=\frac{-\sqrt{(1-uu)}}{u}$. Pro altero membro $\frac{-du}{uu\sqrt{(1-uu)}}\cdot\sqrt{(1-muu)}=\sqrt{(1-muu)}\,.\,d\,.\frac{\sqrt{(1-uu)}}{u}$ habebimus

$$\int\frac{-du}{uu\sqrt{(1-uu)}}\cdot\sqrt{(1-muu)}=\frac{\sqrt{(1-uu)(1-muu)}}{u}+\int\frac{mdu\sqrt{(1-uu)}}{\sqrt{(1-muu)}}$$

hincque $CN-AM=-\frac{a}{\sqrt{m}}\left(-\frac{\sqrt{(1-uu)}}{u}+\frac{\sqrt{(1-uu)(1-muu)}}{u}+\int\frac{mdu\sqrt{(1-uu)}}{\sqrt{(1-muu)}}\right)$.

At si u evanescit, fit $\sqrt{(1-muu)}=1-\frac{1}{2}\cdot muu$, et pars integrata sponte evanescit, ita ut jam sit

$$CN-AM=-a\sqrt{m}\int\frac{du\sqrt{(1-uu)}}{\sqrt{(1-muu)}},$$

quod integrari debet a termino $u=1$ usque ad $u=0$; sin autem integremus ab $u=0$ usque ad $u=1$, habebimus

$$CN-AM=a\sqrt{m}\int\frac{du\sqrt{(1-uu)}}{\sqrt{(1-muu)}},$$

cujus valor per rectificationem sectionis conicae assignari potest, uti constat. Quemadmodum revera est differentia inter asymtotam et arcum hyperbolae vide Nov. Comm. T. VIII pag. 134 cas. II.

(N. Fuss.)

Erit enim $CN - AM = aC\sqrt{m} - \frac{am}{m-1}(1 - u\sqrt{m})\Pi$, ubi Π est arcus a vertice sumtus sectionis conicae, cujus semiparameter $= 1$ et semiaxis transversus $= a$, pro terminis integrationis supra stabilitis.

(J. A. Euler.)

Haec formula
$$\int \frac{du\sqrt{(1-uu)}}{\sqrt{(1-muu)}}$$
duplici modo in seriem evolvi potest.

I. Modus. Cum sit $(1 - muu)^{-\frac{1}{2}} = 1 + \frac{1}{2}muu + \frac{1.3}{2.4}m^2u^4 + \frac{1.3.5}{2.4.6}m^3u^6 +$ etc. et

$$\int u^{\lambda+2}du\sqrt{(1-uu)} = \frac{\lambda+1}{\lambda+4}\int u^{\lambda}du\sqrt{(1-uu)} - \frac{1}{\lambda+4}u^{\lambda+1}(1-uu)^{\frac{3}{2}},$$

ubi postremum membrum ab $u = 0$ usque ad $u = 1$ sumtum evanescit; quare cum sit $\int du\sqrt{(1-uu)} = \frac{\pi}{4}$, erit

$$\int uudu\sqrt{(1-uu)} = \frac{1}{4}\cdot\frac{\pi}{4}$$
$$\int u^4du\sqrt{(1-uu)} = \frac{1.3}{4.6}\cdot\frac{\pi}{4}$$
$$\int u^6du\sqrt{(1-uu)} = \frac{1.3.5}{4.6.8}\cdot\frac{\pi}{4}$$
etc.

consequenter fit $CN - AM = \frac{\pi a\sqrt{m}}{4}\left(1 + \frac{1.1}{2.4}m + \frac{1.1}{2.4}\cdot\frac{3.3}{4.6}m^2 + \frac{1.1}{2.4}\cdot\frac{3.3}{4.6}\cdot\frac{5.5}{6.8}m^3 + \text{etc.}\right)$. Hic notandum si fuerit $m = 1$, fore $CN - AM = a\sqrt{m}\int du$, ut fieri debeat $CN - AM = a$, unde sequitur fore

$$1 = \frac{\pi}{4}\left(1 + \frac{1.1}{2.4} + \frac{1.1}{2.4}\cdot\frac{3.3}{4.6} + \text{ etc.}\right)$$

ideoque haec series $= \frac{4}{\pi}$. Alter casus, quo $n = 0$ et $m = 0$, manifesto prodit $CN - AM = 0$.

II. Modus. Ponatur $u = \sin\varphi$, ita ut integrari oporteat a $\varphi = 0$ usque ad $\varphi = \frac{\pi}{2}$, et habebimus

$$CN - AM = a\sqrt{m}\int\frac{d\varphi\cos^2\varphi}{\sqrt{(1-m\sin^2\varphi)}} = \frac{a\sqrt{m}}{\sqrt{(4-2m)}}\int\frac{d\varphi(1+\cos 2\varphi)}{\sqrt{\left(1+\frac{m}{2-m}\cos 2\varphi\right)}}.$$

Sit nunc brevitatis gratia $\frac{m}{2-m} = k = \frac{nn}{2+nn}$ et

$$CN - AM = a\sqrt{\frac{1}{2}k}\int d\varphi(1+\cos 2\varphi)(1+k\cos 2\varphi)^{-\frac{1}{2}}.$$

Jam vero est $(1 + k\cos 2\varphi)^{-\frac{1}{2}} = 1 - \frac{1}{2}k\cos 2\varphi + \frac{1.3}{2.4}kk\cos^2 2\varphi - \frac{1.3.5}{2.4.6}k^3\cos^3 2\varphi +$ etc. Porro notetur esse

$$\cos^2 2\varphi = \frac{1}{2} + \frac{1}{2}\cos 4\varphi$$
$$\cos^3 2\varphi = \frac{3}{4}\cos 2\varphi + \frac{1}{4}\cos 6\varphi$$
$$\cos^4 2\varphi = \frac{1.3}{2.4} + \frac{1}{2}\cos 4\varphi + \frac{1}{8}\cos 8\varphi$$
$$\cos^5 2\varphi = \frac{5}{8}\cos 2\varphi + \frac{5}{16}\cos 6\varphi + \frac{1}{16}\cos 10\varphi$$

$$\cos^6 2\varphi = \frac{1.3.5}{2.4.6} + \text{ etc.}$$

$$\cos^7 2\varphi = 2 \cdot \frac{1.3.5.7}{2.4.6.8} \cos 2\varphi + \text{ etc.}$$

$$\cos^8 2\varphi = \frac{1.3.5.7}{2.4.6.8} + \text{ etc.}$$

Deinde notetur esse $\int d\varphi \cos 2\lambda\varphi = \frac{1}{2\lambda} \sin 2\lambda\varphi$, quod casu $\varphi = 90^0$ fit $= 0$; unde patet in evolutione omnes terminos $\sin 2\lambda\varphi$ continentes omitti posse, unde nostra formula summatoria erit

$$\int d\varphi (1 + k \cos 2\varphi)^{-\frac{1}{2}} = \int d\varphi \left(1 + \frac{1.3}{2.4} \cdot \frac{1}{2} kk + \frac{1.3.5.7}{2.4.6.8} \cdot \frac{1.3}{2.4} k^4 + \text{ etc.}\right)$$

$$\int d\varphi \cos 2\varphi (1 + k \cos 2\varphi)^{-\frac{1}{2}} = \int d\varphi \left(-\frac{1}{2} \cdot \frac{1}{2} k - \frac{1.3.5}{2.4.6} \cdot \frac{1.3}{2.4} k^3 - \frac{1.3.5.7.9}{2.4.6.8.10} \cdot \frac{1.3.5}{2.4.6} k^5 - \text{ etc.}\right)$$

consequenter $CN - AM =$

$$\frac{1}{2} a\pi \sqrt{\frac{1}{2} k} \left(1 - \frac{1}{4} k + \frac{1.3}{4.4} kk - \frac{1.3.5}{4.4.8} k^3 + \frac{1.3.5.7}{4.4.8.8} k^4 - \frac{1.3.5.7.9}{4.4.8.8.12} k^5 + \text{ etc.}\right)$$

casu ergo, quo $n = \infty$, fit $k = 1$, hic vero valor fieri debet $= a$, unde sequitur

$$\frac{2\sqrt{2}}{\pi} = \left(1 - \frac{1}{4} + \frac{1.3}{4.4} - \frac{1.3.5}{4.4.8} + \frac{1.3.5.7}{4.4.8.8} - \text{ etc.}\right).$$

Proposita autem vicissim hac serie, ejus valor ita investigari potest. Fiat $k = zz$ et ponatur

$$s = 1 + \frac{1.3}{4.4} z^4 + \frac{1.3}{4.4} \cdot \frac{5.7}{8.8} z^8 + \text{ etc. et}$$

$$t = \frac{1}{4} z^2 + \frac{1.3.5}{4.4.8} z^6 + \frac{1.3.5.7.9}{4.4.8.8.12} z^{10} + \text{ etc.}$$

ita ut $s - t$ praebeat nostram seriem. Hinc erit

$$\frac{ds}{dz} = \frac{1.3}{4} z^3 + \frac{1.3.5.7}{4.4.8} z^7 + \text{ etc.}$$

$$\frac{d.tz}{dz} = \frac{1.3}{4} z^2 + \frac{1.3.5.7}{4.4.8} z^6 + \text{ etc.} = \frac{ds}{zdz}$$

hinc $zdt + tdz = \frac{ds}{z}$. Porro

$$\frac{d.sz}{dz} = 1 + \frac{1.3.5}{4.4} z^4 + \frac{1.3.5.7.9}{4.4.8.8} z^8 + \text{ etc.}$$

$$\frac{d.tzz}{dz} = 1 . z^3 + \frac{1.3.5}{4.4} z^7 + \text{ etc.} = \frac{z^3.d.sz}{dz}$$

hinc $zzdt + 2tzdz = z^4 ds + sz^3 dz$. En ergo has duas aequationes, ex quibus eliminando ds reperitur

$$s = \frac{(1 - z^4)\, dt}{zdz} + \frac{(2 - z^4)\, t}{zz}$$

unde

$$\left.\begin{aligned} ds = \frac{ddt(1 - z^4)}{zdz} - dt\left(\frac{1}{zz} + 3zz\right) + t\left(-\frac{4}{z^3} - 2z\right) dz \\ + dt\left(\frac{2}{zz} - zz\right) \end{aligned}\right\} = zzdt + tzdz;$$

unde resultat haec aequatio

$$0 = zzddt(1-z^4) + zdzdt(1-5z^4) - tdz^2(4+3z^4),$$

unde si inventum fuerit t, tunc erit $s = \frac{(1-z^4)\,dt}{zdz} + \frac{(2-z^4)\,t}{zz}$.

Illa autem aequatio ad differentialem primi gradus reducitur ponendo $t = e^{\int \nu dz}$, dum erit $dt = e^{\int \nu dz}\,\nu dz$ et $ddt = e^{\int \nu dz}(d\nu dz + \nu\nu dz^2)$, quibus substitutis reperitur

$$zzd\nu(1-z^4) + zz\nu\nu dz(1-z^4) + \nu z dz(1-5z^4) - dz(4+3z^4) = 0.$$

Statuatur $\nu = \frac{q}{z(1-z^4)}$, erit $d\nu = \frac{dq}{z(1-z^4)} - \frac{qdz(1-5z^4)}{zz(1-z^4)^2}$; quibus substitutis nanciscimur

$$dq + \frac{qqdz}{z(1-z^4)} - \frac{dz(4+3z^4)}{z} = 0.$$

Lemma. Notetur haec reductio $\int z^{m+n-1}dz(1-z^n)^{k-1} = \frac{m}{m+kn}\int z^{m-1}dz(1-z^n)^{k-1}$, si integretur a $z=0$ usque $z=1$.

Alia methodus eandem seriem investigandi. Quaeratur separatim series

$$s = 1 + \frac{1.3}{4.4}kk + \frac{1.3.5.7}{4.4.8.8}k^4 + \text{ etc.}$$

$$\text{et } t = \frac{1}{4}k + \frac{1.3.5}{4.4.8}k^3 + \frac{1.3.5.7.9}{4.4.8.8.12}k^5 + \text{ etc.}$$

Pro priore consideretur formula

$$(1-kkz^4)^{-\frac{1}{4}} = 1 + \frac{1}{4}kkz^4 + \frac{1.5}{4.8}k^4z^8 + \frac{1.5.9}{4.8.12}k^6z^{12} + \text{ etc.}$$

hinc erit

$$\int dp(1-kkz^4)^{-\frac{1}{4}} = \int dp + \frac{1}{4}kk\int z^4dp + \frac{1.5}{4.8}k^4\int z^8dp + \text{ etc.}$$

Nunc fiat $\int z^4dp = \frac{3}{4}\int dp$, et $\int z^8dp = \frac{7}{8}\int z^4dp$, et $\int z^{12}dp = \frac{11}{12}\int z^8dp$, erit

$$s = \frac{\int dp(1-kkz^4)^{-\frac{1}{4}}}{\int dp} = 1 + \frac{1.3}{4.4}kk + \frac{1.3.5.7}{4.4.8.8}k^4 + \text{ etc.}$$

Ex superiore lemmate habemus $\int \frac{z^{m+3}dz}{(1-z^4)^{\frac{3}{4}}} = \frac{m}{m+1}\int \frac{z^{m-1}dz}{(1-z^4)^{\frac{3}{4}}}$, unde fit

$$\int \frac{z^6dz}{(1-z^4)^{\frac{3}{4}}} = \frac{3}{4}\int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}}, \text{ deinde } \int \frac{z^{10}dz}{(1-z^4)^{\frac{3}{4}}} = \frac{7}{8}\int \frac{z^6dz}{(1-z^4)^{\frac{3}{4}}}.$$

Unde patet sumi debere $dp = \frac{zzdz}{(1-z^4)^{\frac{3}{4}}}$, consequenter erit

$$s = \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}(1-kkz^4)^{\frac{1}{4}}} : \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}}$$

Pro altera serie $t = \frac{1}{4}k + \frac{1.3.5}{4.4.8}k^3 + \frac{1.3.5.7.9}{4.4.8.8.12}k^5 + \text{ etc.}$ Consideretur

$$\frac{(1-kkz^4)^{-\frac{1}{4}} - 1}{kzz} = \frac{1}{4}kz^2 + \frac{1.5}{4.8}k^3z^6 + \frac{1.5.9}{4.8.12}k^5z^{10} + \text{ etc.}$$

Fiat $\int z^6dp = \frac{3}{4}\int z^2dp$, $\int z^{10}dp = \frac{7}{8}\int z^6dp$ etc. Hinc $dp = \frac{dz}{(1-z^4)^{\frac{3}{4}}}$, unde sequitur

$$t = \int \frac{dz\left((1-kkz^4)^{-\frac{1}{4}} - 1\right)}{kzz(1-z^4)^{\frac{3}{4}}} : \int \frac{dz}{(1-z^4)^{\frac{3}{4}}}.$$

Hinc autem neutiquam patet quomodo haec series commodius exprimi possit.

A. m. T. I. p. 258 — 266.

101.

(N. Fuss.)

Si trianguli latera fuerint

$$a = rs(qq + tt), \qquad b = qt(rr + ss), \qquad c = (qr + st)(rt - qs)$$

erit area $= qrst(qr + st)(rt - qs)$.

At si quadrilateri circulo inscripti latera fuerint

$$a = f - pqr, \qquad b = f - pst, \qquad c = f - qsu, \qquad d = f - rtu$$

existente $f = \frac{1}{2}p(qr + st) + \frac{1}{2}u(qs + rt)$, hic scilicet est f semisumma laterum; tum vero area quadrilateri erit

$$= pqrstu.$$

A. m. T. I. p. 326.

102.

(N. Fuss.)

Theorema geometricum. (Fig. 65.) Si quatuor puncta A, B, C, D utcunque fuerint sita, eorumque bina jungantur sex lineis rectis AB, AC, AD, BC, BD, CD, inter has sex lineas talis est relatio, ut sequens aequatio locum obtineat:

$$\begin{aligned} & AB^2 . CD^2 (AB^2 + CD^2) - AB^2 . CD^2 (BC^2 + BD^2 + AC^2 + AD^2) + BC^2 . BD^2 . CD^2 \\ & + AC^2 . BD^2 (AC^2 + BD^2) - AC^2 . BD^2 (AB^2 + AD^2 + BC^2 + CD^2) + AC^2 . AD^2 . CD^2 \\ & + BC^2 . AD^2 (BC^2 + AD^2) - BC^2 . AD^2 (AB^2 + AC^2 + BD^2 + CD^2) + AB^2 . AD^2 . BD^2 \\ & \qquad + AB^2 . AC^2 . BC^2 = 0 \end{aligned}$$

ubi in tertiae columnae quovis termino tres rectae triangulum constituentes conjunguntur, in prioribus autem columnis ratio compositionis est manifesta.

Demonstratio. Sint latera $AB = a$, $BC = b$, $CD = c$, $AD = d$ et diagonales $AC = p$ et $BD = q$. Considerentur anguli x et y, et ex triangulo ABC erit $\cos y = \frac{aa + pp - bb}{2ap} = \alpha$, et ex triangulo ACD erit

$$\cos x = \frac{dd + pp - cc}{2dp} = \beta.$$

At vero ex triangulo ABD erit $\qquad \cos(x + y) = \frac{aa + dd - qq}{2ad} = \gamma$;

hinc ergo erit

$$\sin\tfrac{1}{2}y = \sqrt{\frac{1-\alpha}{2}}, \quad \cos\tfrac{1}{2}y = \sqrt{\frac{1+\alpha}{2}}, \quad \sin\tfrac{1}{2}x = \sqrt{\frac{1-\beta}{2}} \quad \text{et} \quad \cos\tfrac{1}{2}x = \sqrt{\frac{1+\beta}{2}}$$

unde fiet $$\sin\left(\frac{x+y}{2}\right) = \sqrt{\frac{(1-\beta)(1+\alpha)}{4}} + \sqrt{\frac{(1-\alpha)(1+\beta)}{4}}$$
et $$\cos\left(\frac{x+y}{2}\right) = \sqrt{\frac{(1+\alpha)(1+\beta)}{4}} - \sqrt{\frac{(1-\alpha)(1-\beta)}{4}}$$

At vero ex tertia aequatione $\sin\left(\frac{x+y}{2}\right) = \sqrt{\frac{1-\gamma}{2}}$ et $\cos\left(\frac{x+y}{2}\right) = \sqrt{\frac{1+\gamma}{2}}$, unde nascuntur hae duae aequationes

$$\sqrt{(1-\beta)(1+\alpha)} + \sqrt{(1-\alpha)(1+\beta)} = \sqrt{2(1-\gamma)} \quad \text{et} \quad \sqrt{(1+\alpha)(1+\beta)} - \sqrt{(1-\alpha)(1-\beta)} = \sqrt{2(1+\gamma)}.$$

Sumatur prioris quadratum et reperietur $\sqrt{(1-\alpha\alpha)(1-\beta\beta)} = \alpha\beta - \gamma$, hincque denuo sumtis quadratis: $1 - \alpha\alpha - \beta\beta - \gamma\gamma + 2\alpha\beta\gamma = 0$. Hic igitur tantum opus est, ut pro α, β, γ valores substituantur, scilicet

$$\alpha = \frac{aa+pp-bb}{2ap}, \quad \beta = \frac{dd+pp-cc}{2dp}, \quad \gamma = \frac{aa+dd-qq}{2ad};$$

quo facto et per denominatorem $4aaddpp$ multiplicando, si termini in ordinem redigantur, erit

$$\begin{aligned} &aacc(aa+cc) - aacc(bb+dd+pp+qq) + aabbpp \\ &+ bbdd(bb+dd) - bbdd(aa+cc+pp+qq) + ccddpp \\ &+ ppqq(pp+qq) - ppqq(aa+bb+cc+dd) + aaddqq \\ &+ bbccqq = 0. \end{aligned}$$

Multo brevius autem hoc negotium fieri potest, posito $x+y=z$; erit $\cos z = \cos x \cos y - \sin x \sin y$, ergo $\sin x \sin y = \cos x \cos y - \cos z$ et sumtis quadratis $\sin^2 x \sin^2 y = \cos^2 x \cos^2 y - 2\cos x \cos y \cos z + \cos^2 z$

at est $$\sin^2 x \sin^2 y = 1 - \cos^2 x - \cos^2 y + \cos^2 x \cos^2 y,$$

ideoque $1 - \cos^2 x - \cos^2 y + 2\cos x \cos y \cos z - \cos^2 z = 0$, hoc est

$$1 - \alpha\alpha - \beta\beta - \gamma\gamma + 2\alpha\beta\gamma = 0;$$

reliqua manent, ut ante. Cum igitur sit

$$\cos x = \frac{dd+pp-cc}{2dp} = \frac{X}{2dp}, \quad \cos y = \frac{aa+pp-bb}{2ap} = \frac{Y}{2ap} \quad \text{et} \quad \cos z = \frac{aa+dd-qq}{2ad} = \frac{Z}{2ad},$$

hincque fiet $1 - \frac{XX}{4ddpp} - \frac{YY}{4aapp} - \frac{ZZ}{4aadd} + \frac{XYZ}{4aaddpp} = 0$ et per $4aaddpp$ multiplicando

$$4aaddpp - aaXX - ddYY - ppZZ + XYZ = 0.$$

Sumto nunc (Fig. 66) in triangulo puncto quocunque, ex quo ad singulos trianguli angulos ducantur rectae * x, y, z, erit

$$\begin{aligned} &aaxx(aa+xx) - aaxx(bb+cc+yy+zz) + aabbcc \\ &+ bbyy(bb+yy) - bbyy(aa+cc+xx+zz) + aayyzz \\ &+ cczz(cc+zz) - cczz(aa+bb+xx+yy) + bbxxzz \\ &+ ccxxyy = 0 \end{aligned}$$

quae ita disponi potest

$$\begin{aligned} &aax^4 - xxyy(aa+bb-cc) - aaxx(bb+cc-aa) + aabbcc \\ &+ bby^4 - xxzz(aa+cc-bb) - bbyy(aa+cc-bb) \\ &+ ccz^4 - yyzz(bb+cc-aa) - cczz(aa+bb-cc) = 0. \end{aligned}$$

A. m. T. I. p. 345. 346.

103.

(N. Fuss.)

* Problema. (Fig. 67.) Angulum ACB, sive arcum AB in n partes aequales proxime dividere.

Solutio. In AC producta capiatur $Ca = \frac{n-2}{2n-1} AC$; deinde in radio CB capiatur $Cb = \frac{n-2}{n+1} CB$; tum per puncta a, b agatur recta arcum secans in O, eritque BO proxime $\frac{1}{n} AB$. Ita si angulus debeat trisecari, ob $n = 3$, erit $Ca = \frac{1}{5} AC$ et $Cb = \frac{1}{4} CB$.

A. m. T. II. p. 26.

104.

(N. Fuss.)

* Theorema. (Fig. 68.) Si arcus circuli quincunque ab in duobus punctis p et q utcunque secetur, semper erit

$$\sin n \,.\, aq \,.\, \sin n \,.\, bp = \sin n \,.\, ap \,.\, \sin n \,.\, bq + \sin n \,.\, ab \,.\, \sin n \,.\, pq$$

ubi pro n numerum quemcunque accipere licet.

Hinc si fuerit n numerus valde parvus, erit

$$aq \,.\, bp = ap \,.\, bq + ab \,.\, pq.$$

A. m. T. II. p. 132.

III. Analysis.

105.

(Lexell.)

Criterium pro dignoscendis radicibus rationalibus aequationum cubicarum.

Quum aequationis cubicae ternae radices ita exprimantur:

$$I \,.\, x = p + \sqrt[3]{q} + \sqrt[3]{r}, \quad II \,.\, x = p - \frac{(1+\sqrt{-3})}{2}\sqrt[3]{q} - \frac{(1-\sqrt{-3})}{2}\sqrt[3]{r}, \quad III \,.\, x = p - \frac{(1-\sqrt{-3})}{2}\sqrt[3]{q} - \frac{(1+\sqrt{-3})}{2}\sqrt[3]{r}$$

hae tres formae rationales esse nequeunt, nisi $\sqrt[3]{q}$ et $\sqrt[3]{r}$ sequenti modo exhibere liceat: $\sqrt[3]{q} = s + t\sqrt{-3}$ et $\sqrt[3]{r} = s - t\sqrt{-3}$; tum enim fiet

$$I \,.\, x = p + 2s, \quad II \,.\, x = p - s - 3t, \quad III \,.\, x = p - s + 3t;$$

tum autem ipsae litterae q et r similes formas habebunt; erit scilicet $q = u + \nu\sqrt{-3}$ et $r = u - \nu\sqrt{-3}$. His praenotatis, consideremus aequationem cubicam: $x^3 = 3fx + 2g$, cujus radicem constat esse

$$x = \sqrt[3]{(g + \sqrt{(gg - f^3)})} + \sqrt[3]{(g - \sqrt{(gg - f^3)})}.$$

Ut ergo omnes tres radices sint rationales, ob $g \pm \sqrt{(gg - f^3)} = u \pm \nu\sqrt{-3}$, evidens est esse debere $\sqrt{(gg - f^3)} = \nu\sqrt{-3}$, ideoque $\nu = \sqrt{\frac{-gg + f^3}{3}}$ et $f^3 - gg = 3\nu\nu$, sive $\frac{f^3 - gg}{3} = \square$; unde concludimus, quoties $\frac{f^3 - gg}{3}$ fuerit quadratum, etiam omnes tres radices fore rationales.

Exemplum. Sint radices $I \,.\, x = 3$, $II \,.\, x = 7$ et $III \,.\, x = 10$, unde aequatio resultat $x^3 - 79x + 210 = 0$, sive $x^3 = 79x - 210$, ubi $f = \frac{79}{3}$ et $g = -105$, hinc $f^3 = \frac{493039}{27}$ et $gg = 11025$, ergo $f^3 - gg = \frac{195364}{27}$, consequenter $\frac{f^3 - gg}{3} = \frac{195364}{81}$ et $\sqrt{\frac{f^3 - gg}{3}} = \frac{442}{9}$, unde fit $\nu = \frac{442}{9}$ et $u = -105$.

Hoc idem autem in genere ita ostenditur: Sint ternae radices $x = a$, $x = b$, $x = -a - b$, ita ut aequatio sit $x^3 = (a^2 + ab + b^2)x - ab(a + b)$. Hinc fit

$$f = \frac{a^2 + ab + b^2}{3} \text{ et } g = \frac{-ab(a+b)}{2}, \quad f^3 = \frac{a^6 + 3a^5b + 6a^4b^2 + 7a^3b^3 + 6a^2b^4 + 3ab^5 + b^6}{27}$$

et
$$gg = \frac{a^4b^2 + 2a^3b^3 + a^2b^4}{4}, \quad \text{ergo} \quad \frac{f^3 - g^2}{3} = \frac{4a^6 + 12a^5b - 3a^4b^2 - 26a^3b^3 - 3a^2b^4 + 12ab^5 + 4b^6}{81.4}$$

hinc
$$\sqrt{\frac{f^3 - g^2}{3}} = \frac{2a^3 + 3a^2b - 3ab^2 - 2b^3}{9.2} = \frac{(a-b)(2a+b)(a+2b)}{18}.$$

Observatio I. Quum $f^3 - g^2$ debeat esse triplum quadratum scilicet $3\nu\nu$, sive $f^3 = gg + 3\nu\nu$, certum est hoc fieri non posse, nisi ipse numerus f jam habeat formam similem $mm + 3nn$, unde sequitur, si numerus f in suos factores primos resolvatur, unumquemque fore formae $6\alpha + 1$, cujusmodi numeri primi sunt 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97; unde statim ac numerus f factores involverit vel 5, vel 11, vel 17 etc. certum est aequationis omnes radices non esse rationales.

Observatio II. Sit igitur f numerus hujus formae $\alpha\alpha + 3\beta\beta$, et quum sit

$$(\alpha\alpha + 3\beta\beta)(pp + 3qq) = (\alpha p \pm 3\beta q)^2 + 3(\alpha q \mp \beta p)^2,$$

erit
$$(\alpha\alpha + 3\beta\beta)^2 = (\alpha\alpha \pm 3\beta\beta)^2 + 3(\alpha\beta \mp \beta\alpha)^2 = (\alpha\alpha - 3\beta\beta)^2 + 3(2\alpha\beta)^2$$

porro
$$(\alpha\alpha + 3\beta\beta)^3 = (\alpha^3 - 3\alpha\beta\beta \pm 6\alpha\beta\beta)^2 + 3(2\alpha\alpha\beta \mp \alpha\alpha\beta \pm 3\beta^3)^2$$

ideoque vel
$$(\alpha\alpha + 3\beta\beta)^3 = (\alpha^3 + 3\alpha\beta\beta)^2 + 3(\alpha\alpha\beta + 3\beta^3)^2$$

vel
$$(\alpha\alpha + 3\beta\beta)^3 = (\alpha^3 - 9\alpha\beta\beta)^2 + 3(3\alpha\alpha\beta - 3\beta^3)^2.$$

Hinc quum sit $f^3 = gg + 3\nu\nu$, erit $g = \pm(\alpha^3 - 9\alpha\beta\beta)$ et $\nu = \pm(3\alpha\alpha\beta - 3\beta^3)$; quare si in aequatione $x^3 = 3fx + 2g$ fuerit $f = \alpha\alpha + 3\beta\beta$ atque insuper $g = \pm(\alpha^3 - 9\alpha\beta\beta)$, tum omnes tres radices erunt rationales, et nisi simul fuerit $f = \alpha\alpha + 3\beta\beta$ atque $g = \pm(\alpha^3 - 9\alpha\beta\beta)$, omnes tres radices rationales esse non possunt.

Observatio III. Sin autem f et g tales habuerint formas, ut sit $x^3 = 3(\alpha\alpha + 3\beta\beta)x + 2\alpha(\alpha\alpha - 9\beta\beta)$, radices certe erunt rationales, quippe quae erunt $x = 2\alpha$, $x = -\alpha + 3\beta$, et $x = -\alpha - 3\beta$. Hinc igitur veritas nostri criterii ita est stabilita, ut non solum praesentia criterii tres radices rationales indicet, sed etiam rationalitas radicum ipsum hoc criterium involvat.

Observatio IV. Videamus autem quoque, quomodo hoc criterium ad formam generalem aequationum cubicarum applicari debeat. Proposita igitur sit forma generalis $z^3 + Pz^2 + Qz + R = 0$; primo ergo ad formam praecedentem revocetur ponendo $z = x - \frac{1}{3}P$, et aequatio resultans erit

$$x^3 + (Q - \tfrac{1}{3}P^2)x + \tfrac{2}{27}P^3 - \tfrac{1}{3}PQ + R = 0,$$

sive
$$x^3 = \left(\tfrac{1}{3}P^2 - Q\right)x - \tfrac{2}{27}P^3 + \tfrac{1}{3}PQ - R,$$

unde pro criterio nostro habebimus $f = \frac{1}{9}P^2 - \frac{1}{3}Q$ et $g = -\frac{1}{27}P^3 + \frac{1}{6}PQ - \frac{1}{2}R$, unde fit

$$\frac{f^3 - gg}{3} = \frac{1}{9.36}PPQQ - \frac{1}{81}Q^3 - \frac{1}{81}P^3R + \frac{1}{18}PQR - \frac{1}{12}RR$$

ergo per 324 multiplicando, criterium nostrum postulat, ut sit quadratum sequens forma

$$P^2Q^2 - 4Q^3 - 4P^3R + 18PQR - 27R^2 = \square.$$

A. m. T. I. p. 109. 110.

Continuatio.

Sit cubica aequatio $x^3 = fx + g$ omnes radices habens rationales, quae sint α, β, γ; quia earum summa $=0$, erit $\gamma = -\alpha - \beta$. Jam sint α, β radices hujus aequationis $zz - pz + q = 0$, ubi propterea erit $pp - 4q$ quadratum, haec ergo per $z + p$, hoc est $z + \alpha + \beta$ multiplicata, ipsam propositam producere debet, quae ergo fit $z^3 + (q - pp)z + pq = 0$ sive $z^3 = (pp - q)z - pq$; quocirca erit $f = pp - q$ et $g = -pq$. Quum igitur sit $pp - 4q = \square$, quaeritur quomodo eadem haec conditio per f et g exprimatur, quae est quaestio peculiaris naturae. Multiplicetur $pp - 4q$ per quadratum $p^4 + 2\alpha ppq + \alpha\alpha qq$, ita ut etiam productum

$$p^6 + (2\alpha - 4)p^4q + (\alpha\alpha - 8\alpha)ppqq - 4\alpha\alpha q^3 = \square \quad \text{esse debeat;}$$

manifestum autem est similem formam nasci ex formula $f^3 + \beta gg$; prodit enim $p^6 - 3p^4q + (3 + \beta)ppqq - q^3 = \square$; pro identitate igitur litterae α, β sequenti modo definiuntur: $\alpha = \frac{1}{2}$, $\beta = \frac{-27}{4}$, qui valores etiam postrema membra identica reddunt, ex quo pro rationalitate trium radicum hoc criterium requiritur, ut sit $f^3 - \frac{27}{4}gg = \square$. De hoc autem criterio duo sunt notanda: 1° $f^3 - \frac{27}{4}gg$ debet esse quadratum integrum; 2° hujusmodi aequationes $v^3 = 4fv + 8g$ ad formam simpliciorem ponendo $v = 2x$ debent reduci $x^3 = fx + g$.

A. m. T. I. p. 113. 115.

106.

(Krafft.)

Problema. Si habeatur haec series

$$s = \frac{1}{1} - \frac{1}{1+a} + \frac{1}{1+2a} - \frac{1}{1+3a} + \frac{1}{1+4a} - \text{etc.}$$

ejus quadratum s^2 commode per seriem exprimere. Erit autem

$$s^2 = 1 + \frac{1}{(1+a)^2} + \frac{1}{(1+2a)^2} + \frac{1}{(1+3a)^2} + \text{etc.}$$

$$-\frac{2}{1.(1+a)} - \frac{2}{(1+a)(1+2a)} - \frac{2}{(1+2a)(1+3a)} - \text{etc.} \ldots (= -2A)$$

$$+\frac{2}{1.(1+2a)} + \frac{2}{(1+a)(1+3a)} + \frac{2}{(1+2a)(1+4a)} + \text{etc.} \ldots (= +2B)$$

$$-\frac{2}{1.(1+3a)} - \frac{2}{(1+a)(1+4a)} - \frac{2}{(1+2a)(1+5a)} - \text{etc.} \ldots (= -2C)$$

etc.

Erit vero $$A = \frac{1}{1+a} + \frac{1}{(1+a)(1+2a)} + \frac{1}{(1+2a)(1+3a)} + \text{etc.}$$

sive $$A = \frac{1}{a}\left(\frac{1}{1} - \frac{1}{1+a} + \frac{1}{1+a} - \frac{1}{1+2a} + \frac{1}{1+2a} - \frac{1}{1+3a} + \text{etc.}\right)$$

ergo $A = \frac{1}{a} \cdot 1$. Similiter

$$B = \frac{1}{2a}\left(\frac{1}{1} - \frac{1}{1+2a} + \frac{1}{1+a} - \frac{1}{1+3a} + \frac{1}{1+2a} - \frac{1}{1+4a} + \frac{1}{1+3a} - \text{etc.}\right)$$

ergo $B = \frac{1}{2a}\left(\frac{1}{1} + \frac{1}{1+a}\right)$. Eodem modo

$$C=\frac{1}{3a}\left(\frac{1}{1}-\frac{1}{1+3a}+\frac{1}{1+a}-\frac{1}{1+4a}+\frac{1}{1+2a}-\frac{1}{1+5a}+\frac{1}{1+3a}-\frac{1}{1+6a}+\ \text{etc.}\right)$$

ergo $C=\frac{1}{3a}\left(\frac{1}{1}+\frac{1}{1+a}+\frac{1}{1+2a}\right)$ et ita porro. Quocirca fiet

$$s^2=1+\frac{1}{(1+a)^2}+\frac{1}{(1+2a)^2}+\frac{1}{(1+3a)^2}+\ \text{etc.}$$

$$-\frac{2}{a}\cdot 1+\frac{2}{2a}\left(\frac{1}{1}+\frac{1}{1+a}\right)-\frac{2}{3a}\left(\frac{1}{1}+\frac{1}{1+a}+\frac{1}{1+2a}\right)+\ \text{etc.}$$

cujus partis posterioris valor ita investigetur: Ponatur

$$z=-\frac{2x}{a}\cdot 1+\frac{2x^2}{2a}\left(\frac{1}{1}+\frac{1}{1+a}\right)-\frac{2x^3}{3a}\left(\frac{1}{1}+\frac{1}{1+a}+\frac{1}{1+2a}\right)+\ \text{etc.}$$

erit
$$\frac{dz}{dx}=-\frac{2}{a}\cdot 1+\frac{2x}{a}\left(1+\frac{1}{1+a}\right)-\frac{2x^2}{a}\left(1+\frac{1}{1+a}+\frac{1}{1+2a}\right)+\ \text{etc.}$$

$$\frac{xdz}{dx}=\qquad -\frac{2x}{a}\cdot 1\qquad +\frac{2x^2}{a}\left(1+\frac{1}{1+a}\right)\qquad -\ \text{etc.}$$

adeoque
$$\frac{(1+x)\,dz}{dx}=-\frac{2}{a}+\frac{2x}{a(1+a)}-\frac{2x^2}{a(1+2a)}+\ \text{etc.}$$

Ponatur $x=y^a$, ut habeatur

$$\frac{(1+y^a)\,dz}{ay^{a-1}\,dy}=-\frac{2}{a}+\frac{2y^a}{a(1+a)}-\frac{2y^{2a}}{a(1+2a)}+\ \text{etc.}$$

seu
$$\frac{(1+y^a)\,dz}{y^{a-2}dy}=-\frac{2y}{1}+\frac{2y^{a+1}}{1+a}-\frac{2y^{2a+1}}{1+2a}+\ \text{etc.}$$

$$\frac{d\cdot\frac{(1+y^a)\,dz}{y^{a-2}dy}}{dy}=-2+2y^a-2y^{2a}+2y^{3a}-\ \text{etc.}=\frac{-2}{1+y^a}$$

ergo
$$\frac{(1+y^a)\,dz}{y^{a-2}dy}=-2\int\frac{dy}{1+y^a}\quad \text{et}\quad dz=\frac{-2y^{a-2}dy}{1+y^a}\int\frac{dy}{1+y^a},$$ consequenter

$$z=-2\int\frac{y^{a-2}dy}{1+y^a}\int\frac{dy}{1+y^a}$$

Posito ergo $y=1$ erit quadratum quaesitum

$$s^2=1+\frac{1}{(1+a)^2}+\frac{1}{(1+2a)^2}+\ \text{etc.}\ +z.$$

Hic vero occurrit casus memorabilis, quando $a=2$, ideoque

$$s=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ \text{etc.}\ =\frac{\pi}{4}.$$

tum autem fit
$$z=-2\int\frac{dy}{1+y^2}\cdot\int\frac{dy}{1+y^2}=-(\text{Arc. tang}\,y)^2=-\frac{\pi^2}{16},$$

unde tandem oritur
$$s^2=\frac{\pi^2}{16}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \text{etc.}\ -\frac{\pi^2}{16}$$

adeoque
$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \text{etc.}\ =\frac{\pi^2}{8}.$$

Sin autem fuerit $a=1$, ideoque

$$s=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \text{etc.}\ =\log.2$$

tum
$$z=-2\int\frac{dy}{y(1+y)}\int\frac{dy}{1+y}=-2\int\frac{dy\log.(1+y)}{y(1+y)}$$

et ponendum erit post integrationem $y=1$, eritque

$$s^2=(\log.2)^2=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ \text{etc.}\ -2\int\frac{dy\log.(1+y)}{y(1+y)}.$$

A. m. T. I. p. 162 — 164.

107.

(N. Fuss.)

Theorema. Proposita serie potestatum quacunque

$$P = 1 + x^\alpha + x^\beta + x^\gamma + x^\delta + \text{ etc.}$$

ejusque sumatur potestas exponentis λ, nempe P^λ, in qua evoluta occurrat terminus $[n]\,x^n$, ejus coëfficiens $[n]$ ita pendebit ab aliquibus praecedentium, ut sit

$$n\,[n] = {}^{+\,\lambda\alpha}_{-\,(n-\alpha)}\,[n-\alpha]\;{}^{+\,\lambda\beta}_{-\,(n-\beta)}\,[n-\beta]\;{}^{+\,\lambda\gamma}_{-\,(n-\gamma)}\,[n-\gamma]\;{}^{+\ \text{etc.}}_{-\ \text{etc.}}$$

quae expressio eousque est continuanda, quamdiu numeri $n-\alpha$, $n-\beta$, $n-\gamma$, etc. non fiunt negativi.

Demonstratio. Ponatur $P^\lambda = S$, erit $lS = \lambda lP$ et $\frac{dS}{S} = \frac{\lambda dP}{P}$, hincque $PdS = \lambda SdP$, quae aequalitas ita repraesentetur $P \cdot \frac{xdS}{dx} = \lambda S \cdot \frac{xdP}{dx}$. Cum igitur sit $P = 1 + x^\alpha + x^\beta + x^\gamma +$ etc. erit

$$\frac{xdP}{dx} = \alpha x^\alpha + \beta x^\beta + \gamma x^\gamma + \delta x^\delta + \text{ etc.}$$

Jam in serie S occurrat terminus $[n]\,x^n$, praeter quem considerentur eae potestates, quae per $\frac{xdP}{dx}$ multiplicatae producere possunt potestatem x^n, qui termini ita repraesententur

$$S = \ldots\ldots [n]\,x^n\,[n-\alpha]\,x^{n-\alpha}\,[n-\beta]\,x^{n-\beta} \text{ etc.}$$

Hinc ergo erit

$$\lambda S \frac{xdP}{dx} = \lambda\alpha\,[n-\alpha]\,x^n + \lambda\beta\,[n-\beta]\,x^n + \lambda\gamma\,[n-\gamma]\,x^n + \text{ etc.}$$

Deinde cum ex iisdem terminis sit

$$\frac{xdS}{dx} = n\,[n]\,x^n + (n-\alpha)\,[n-\alpha]\,x^{n-\alpha} + (n-\beta)\,[n-\beta]\,x^{n-\beta} + \text{ etc.}$$

quae in P ducta, pro potestate x^n praebet sequentes terminos

$$n\,[n]\,x^n + (n-\alpha)\,[n-\alpha]\,x^n + (n-\beta)\,[n-\beta]\,x^n + (n-\gamma)\,[n-\gamma]\,x^n + \text{ etc.}$$

Hi igitur termini x^n utrinque debent poni aequales, unde erit

$$n\,[n] + (n-\alpha)\,[n-\alpha] + (n-\beta)\,[n-\beta] + (n-\gamma)\,[n-\gamma] + \text{ etc.} =$$
$$\lambda\alpha\,[n-\alpha] + \lambda\beta\,[n-\beta] + \lambda\gamma\,[n-\gamma] + \lambda\delta\,[n-\delta] + \text{ etc.}$$

unde conficitur

$$n\,[n] = {}^{\lambda\alpha}_{-\,(n-\alpha)}\,[n-\alpha]\;{}^{+\,\lambda\beta}_{-\,(n-\beta)}\,[n-\beta]\;{}^{+\,\lambda\gamma}_{-\,(n-\gamma)}\,[n-\gamma]\;{}^{+\ \text{etc.}}_{-\ \text{etc.}} \quad \text{Q. E. D.}$$

Corollarium. Cum in serie P exponentes ipsius x sint $0, \alpha, \beta, \gamma, \delta$, etc., in serie $S = P^\lambda$ aliae potestates non occurrunt, nisi quarum exponentes sunt summa λ terminorum hujus seriei $0, \alpha, \beta, \gamma, \delta$, etc. unde si in hac serie S omnes plane potestates ipsius x occurrant, id erit indicio omnes plane numeros reduci posse ad summam λ terminorum istius seriei $0, \alpha, \beta, \gamma, \delta$, etc. At si quaepiam potestas x^n non occurrat, tum ejus coëfficiens $[n]$ aequabitur nihilo. Manifestum autem est, nullum coëfficientem fieri posse negativum.

A. m. T. I. p. 335. 336.

108.

(N. Fuss.)

Teorema. Summa hujus seriei $S = 1 - \frac{1}{2}\left(1 - \frac{1}{2}\right) + \frac{1}{3}\left(1 - \frac{1}{2} + \frac{1}{3}\right) - \frac{1}{4}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) +$ etc. est $S = \frac{\pi\pi}{12} + \frac{1}{2}(l2)^2$.

Demonstratio. Colligantur primo ultimi termini cujusque membri, qui erunt:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{ etc. } = \frac{\pi\pi}{6}.$$

Deinde his terminis exclusis, colligantur denuo termini extremi cujusque membri:

$$-\frac{1}{1.2} - \frac{1}{2.3} - \frac{1}{3.4} - \frac{1}{4.5} - \text{ etc. } = -1.$$

His deletis colligantur denuo ultimi termini, qui sunt

$$\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \text{ etc. } = \frac{1}{2}\left(1 + \frac{1}{2}\right).$$

Simili modo ultimi sequentes erunt

$$-\frac{1}{1.4} - \frac{1}{2.5} - \frac{1}{3.6} - \frac{1}{4.7} - \text{ etc. } = -\frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right).$$

Eodem modo sequentium summa erit $+\frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$ sicque porro. Quare si statuamus

$$t = 1 - \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \text{ etc.}$$

erit $S = \frac{\pi\pi}{6} - t$. Jam istam seriem postremam ita repraesentemus:

$$t = x - \frac{x^2}{2}\left(1 + \frac{1}{2}\right) + \frac{x^3}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \frac{x^4}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \text{ etc.}$$

unde sumto $x = 1$ nostra series t prodit. Nunc autem fiet

$$\frac{dt}{dx} = 1 - x\left(1 + \frac{1}{2}\right) + x^2\left(1 + \frac{1}{2} + \frac{1}{3}\right) - x^3\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \text{ etc.}$$

unde termini primi singulorum terminorum juncti dant

$$1 - x + xx - x^3 + \text{ etc. } = \frac{1}{1+x}.$$

Colligantur porro secundi $\quad -\frac{1}{2}(x - xx + x^3 - x^4 + \text{ etc.}) = \frac{-\frac{1}{2}x}{1+x}.$

Tertii dabunt $\quad +\frac{1}{3}(xx - x^3 + x^4 - x^5 + \text{ etc.}) = \frac{\frac{1}{3}xx}{1+x}.$

Sequentes erunt $\frac{-\frac{1}{4}x^3}{1+x}$, $\frac{+\frac{1}{5}x^4}{1+x}$, etc. Quamobrem erit

$$\frac{dt}{dx} = \frac{1 - \frac{1}{2}x + \frac{1}{3}xx - \frac{1}{4}x^3 + \text{etc.}}{1+x}$$

fractio, cujus numerator $= \frac{1}{x}l(1+x)$, sicque $\frac{dt}{dx} = \frac{l(1+x)}{x(1+x)}$. Cum igitur sit $\frac{1}{x(1+x)} = \frac{1}{x} - \frac{1}{1+x}$, per partes erit

$$dt = \frac{dx}{x}\, l\,(1+x) - \frac{dx}{1+x}\, l\,(1+x),$$

cujus posterioris membri integrale est $-\frac{1}{2}(l(1+x))^2 = -\frac{1}{2}(l2)^2$.

Pro primo membro $\int \frac{dx}{x}\, l\,(1+x)$, id erit

$$\int \frac{dx}{x}\left(x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{ etc.}\right)$$
$$= x - \frac{xx}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{ etc.}$$

Unde facto $x = 1$, erit haec pars

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{ etc.} = \frac{\pi\pi}{12}.$$

Consequenter habebimus $t = \frac{\pi\pi}{12} - \frac{1}{2}(l2)^2$, ergo summa seriei propositae.

$$S = \frac{\pi\pi}{12} + \frac{1}{2}(l2)^2.$$

Theorema. Sequentis seriei

$$S = 1 - \frac{1}{3}\left(1 - \frac{1}{3}\right) + \frac{1}{5}\left(1 - \frac{1}{3} + \frac{1}{5}\right) - \frac{1}{7}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right) + \text{ etc.}$$

summa erit $S = \frac{3\pi\pi}{32}$.

Demonstratio. Colligantur hic iterum termini postremi singulorum membrorum:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{ etc.} = \frac{\pi\pi}{8}.$$

His deletis reliquorum ultimi termini colligantur, qui sunt

$$-\frac{1}{1.3} - \frac{1}{3.5} - \frac{1}{5.7} - \text{ etc.} = -\frac{1}{2}.1.$$

Sequentium ultimi dant $+\frac{1}{1.5} + \frac{1}{3.7} + \text{ etc.} = \frac{1}{4}\left(1 + \frac{1}{3}\right)$; sequentes erunt

$$-\frac{1}{1.7} - \frac{1}{3.9} - \frac{1}{5.11} - \text{ etc.} = -\frac{1}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right)$$

et ita porro. Hinc erit $S = \frac{\pi\pi}{8} - t$, existente $t = \frac{1}{2}\cdot 1 - \frac{1}{4}\left(1 + \frac{1}{3}\right) + \frac{1}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \text{ etc.}$ Statuatur $t = \frac{xx}{2}\cdot 1 - \frac{x^4}{4}\left(1 + \frac{1}{3}\right) + \frac{x^6}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \text{ etc.}$ fietque

$$\frac{dt}{dx} = x - x^3\left(1 + \frac{1}{3}\right) + x^5\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \text{ etc.}$$

cujus seriei primi termini collecti dant $x - x^3 + x^5 - x^7 + \text{ etc.} = \frac{x}{1+xx}$. Secundi termini:

$$-\frac{x^3}{3} + \frac{x^5}{3} - \frac{x^7}{3} + \text{ etc.} = -\frac{1}{3}\cdot\frac{x^3}{1+xx};$$

sequentes dabunt $+\frac{1}{5}\cdot\frac{x^5}{1+xx}$, sicque erit

$$\frac{dt}{dx} = \frac{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{ etc.}}{1+xx} = \frac{\text{Arc. tang } x}{1+xx}$$

consequenter $t = \int \frac{dx}{1+xx}\cdot\int \frac{dx}{1+xx}$, cujus integrale $t = \frac{1}{2}(\text{Arc. tang } x)^2$. Hinc sumto $x = 1$, erit

$$t = \frac{1}{2}\cdot\frac{\pi\pi}{16} = \frac{\pi\pi}{32}, \quad \text{consequenter} \quad S = \frac{\pi\pi}{8} - \frac{\pi\pi}{32} = \frac{3\pi\pi}{32}.$$

Corollarium. Inventa hac summa si ipsam seriem propositam ita tractemus:

$$S = x - \frac{x^3}{3}\left(1 - \frac{1}{3}\right) + \frac{x^5}{5}\left(1 - \frac{1}{3} + \frac{1}{5}\right) - \text{ etc.}$$

ut fiat

$$\frac{dS}{dx} = 1 - xx\left(1 - \frac{1}{3}\right) + x^4\left(1 - \frac{1}{3} + \frac{1}{5}\right) - \text{ etc.}$$

termini primi dant

$$1 - xx + x^4 - x^6 + \text{ etc. } = \frac{1}{1 + xx}$$

$$\text{secundi: } \frac{1}{3}\cdot\frac{xx}{1+xx}, \text{ tertii: } \frac{1}{5}\cdot\frac{x^4}{1+xx}, \text{ ideoque } \frac{dS}{dx} = \frac{1}{x(1+xx)}\int\frac{dx}{1-xx}.$$

Est vero $\frac{1}{x(1+xx)} = \frac{1}{x} - \frac{x}{1+xx}$, ergo

$$S = \int\frac{dx}{x}\int\frac{dx}{1-xx} - \int\frac{xdx}{1+xx}\int\frac{dx}{1-xx}.$$

Cum igitur sit

$$\int\frac{dx}{1-xx} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \text{ etc.}$$

erit

$$\int\frac{dx}{x}\int\frac{dx}{1-xx} = x + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \frac{x^7}{7^2} + \text{ etc.}$$

Posito ergo $x = 1$, erit

$$\int\frac{dx}{x}\int\frac{dx}{1-xx} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{ etc. } = \frac{\pi\pi}{8}.$$

Quare cum $S = \frac{3\pi\pi}{32}$, erit $\frac{3\pi\pi}{32} = \frac{\pi\pi}{8} - \int\frac{xdx}{1+xx}\int\frac{dx}{1-xx}$; unde sequitur

$$\int\frac{xdx}{1+xx}\int\frac{dx}{1-xx} = \frac{\pi\pi}{32},$$

quem valorem non video quomodo directe erui posset.

Problema. Hanc seriem, secundum numeros primos progredientem,

$$s = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \frac{1}{19} + \frac{1}{23} - \text{ etc.}$$

ubi numeri primi formae $4n-1$ habent signum $+$, reliqui formae $4n+1$ signum $-$, in seriem convergentem convertere.

Solutio. Hoc duplici modo fieri potest. Cum enim primo sit productum

$$\frac{\pi}{4}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{8}{7}\cdot\frac{12}{11}\cdot\frac{12}{13} \text{ etc. } = 1,$$

ubi denominatores sunt numeri primi, numeratores vero pariter pares, unitate vel majores vel minores, sequitur fore

$$s = 1 - \frac{\pi}{4} + \frac{1}{3}\left(1 - \frac{\pi}{4}\right) + \frac{1}{5}\left(\frac{4}{3}\cdot\frac{\pi}{4} - 1\right) + \frac{1}{7}\left(1 - \frac{4.4}{3.5}\cdot\frac{\pi}{4}\right) + \frac{1}{11}\left(1 - \frac{4.4.8}{3.5.7}\cdot\frac{\pi}{4}\right) + \text{ etc.}$$

Deinde cum sit $\frac{\pi}{2}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{14}{13}$ etc. $= 1$, hinc sequitur fore

$$s = \frac{\pi}{2} - 1 - \frac{1}{3}\left(\frac{\pi}{2} - 1\right) - \frac{1}{5}\left(1 - \frac{2}{3}\cdot\frac{\pi}{2}\right) - \frac{1}{7}\left(\frac{2.6}{3.5}\cdot\frac{\pi}{2} - 1\right) - \frac{1}{11}\left(\frac{2.6.6}{3.5.7}\cdot\frac{\pi}{2} - 1\right)$$
$$- \frac{1}{13}\left(1 - \frac{2.6.6.10}{3.5.7.11}\cdot\frac{\pi}{2}\right) - \text{ etc.}$$

quae ambae series manifesto valde convergunt.

Theorema. Posito $\frac{\pi}{4} = q$, si summae sequentium serierum ponantur:

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{ etc. }=Aq.$$

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{ etc. }=2Bqq.$$

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{ etc. }=4Cq^3$$

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\text{ etc. }=8Dq^4$$

etc.

coefficientes ita a se invicem pendent, ut sit

$$A=1,\quad B=1,\quad C=\frac{2AB}{4},\quad D=\frac{2AC+BB}{6},\quad E=\frac{2AD+2BC}{8},\quad F=\frac{2AE+2BD+CC}{10},\quad \text{etc.}$$

unde colliguntur isti valores

$$A=1,\quad B=1,\quad C=\frac{1}{2},\quad D=\frac{1}{3},\quad E=\frac{5}{24},\quad F=\frac{2}{15},\quad G=\frac{61}{720},\quad H=\frac{17}{315}\quad \text{etc.},$$

ubi insuper litterae seorsim per 1, 2, 4, 8, 16, 32 etc. multiplicari debent. Hinc istos numeros ulterius continuavi, quos ergo cum potestatibus ipsius q sequenti modo repraesento. Prior columna valet pro potestatibus imparibus, posterior vero pro paribus:

$Aq=1.q$	$Bq^2=1.2q^2$
$Cq^3=\frac{1}{2}\cdot 4q^3$	$Dq^4=\frac{1}{3}\cdot 8q^4$
$Eq^5=\frac{5}{8.3}\cdot 16q^5$	$Fq^6=\frac{2}{3.5}\cdot 32q^6$
$Gq^7=\frac{61}{16.9.5}\cdot 64q^7$	$Hq^8=\frac{17}{9.5.7}\cdot 128q^8$
$Jq^9=\frac{277}{123.9.7}\cdot 256q^9$	$Kq^{10}=\frac{2.31}{81.5.7}\cdot 512q^{10}$
$Lq^{11}=\frac{19.2659}{256\ 81.25.7}\cdot 1024q^{11}$	$Mq^{12}=\frac{2.691}{81.25.7.11}\cdot 2048q^{12}$
etc.	etc.

Quodsi litterae posterioris columnae ordine dividantur per hos numeros 2.3, 2.15, 2.63, etc. prodeunt meae fractiones $\frac{1}{6}$, $\frac{1}{90}$, $\frac{1}{945}$, $\frac{1}{9450}$, etc.

Supra habuimus haec duo producta

$$\frac{\pi}{4}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{8}{7}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\text{ etc. }=1\quad\text{et}\quad\frac{\pi}{2}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{14}{13}\cdot\text{ etc. }=1;$$

horum prius per posterius divisum dat: $1\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{16}{18}\cdot\frac{20}{22}\cdot$ etc. $=1$. Hae fractiones invertantur et sumantur logarithmi, eritque

$$l\frac{6}{4}+l\frac{6}{8}+l\frac{10}{12}+l\frac{14}{12}+\text{ etc. }=0.$$

Cum igitur sit $l\frac{6}{4}=l\frac{1+\frac{1}{5}}{1-\frac{1}{5}}$, $l\frac{6}{8}=l\frac{1-\frac{1}{7}}{1+\frac{1}{7}}$, etc. evolutis logarithmis semissis dabit hanc aequationem:

$$\left.\begin{array}{l} \frac{1}{5}+\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}+\frac{1}{7}\cdot\frac{1}{5^7} \\ -\frac{1}{7}-\frac{1}{3}\cdot\frac{1}{7^3}-\frac{1}{5}\cdot\frac{1}{7^5}-\frac{1}{7}\cdot\frac{1}{7^7} \\ -\frac{1}{11}-\frac{1}{3}\cdot\frac{1}{11^3}-\frac{1}{5}\cdot\frac{1}{11^5}-\frac{1}{7}\cdot\frac{1}{11^7} \\ +\frac{1}{13}+\frac{1}{3}\cdot\frac{1}{13^3}+\frac{1}{5}\cdot\frac{1}{13^5}+\frac{1}{7}\cdot\frac{1}{13^7} \end{array}\right\}\text{ etc. }=0.$$

Hinc ergo erit

$$\left.\begin{array}{l} \frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\text{ etc.} \\ +\frac{1}{3}\left(\frac{1}{5^3}-\frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{13^3}+\frac{1}{17^3}-\text{ etc.}\right) \\ +\frac{1}{5}\left(\frac{1}{5^5}-\frac{1}{7^5}-\frac{1}{11^5}+\frac{1}{13^5}+\frac{1}{17^5}-\text{ etc.}\right) \\ \text{etc.} \end{array}\right\}=0.$$

Hinc porro

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\text{ etc. }=S=\frac{1}{3}$$
$$+\frac{1}{3}\left(\frac{1}{5^3}-\frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{13^3}+\frac{1}{17^3}-\text{ etc.}\right)$$
$$+\frac{1}{5}\left(\frac{1}{5^5}-\frac{1}{7^5}-\frac{1}{11^5}+\frac{1}{13^5}+\frac{1}{17^5}-\text{ etc.}\right)$$
$$+\frac{1}{7}\left(\frac{1}{5^7}-\frac{1}{7^7}-\frac{1}{11^7}+\frac{1}{13^7}+\frac{1}{17^7}-\text{ etc.}\right)$$
$$\text{etc.}$$

Unde sequitur nostram seriem S aliquantillo majorem esse quam $\frac{1}{3}$.

Observatio. Per similes rationes inveni, si omnes numeri primi in duas partes dividantur unitate differentes, ac pro numeris primis formae $8n+1$ vel $8n+3$ partes majores pro numeratoribus, minores vero pro denominatoribus sumantur; pro his autem numeris $8n-1$ vel $8n-3$ minores pro numeratoribus et majores pro denominatoribus sumantur, productum omnium harum fractionum erit $=1$, hoc est

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{10}{9}\text{ etc. }=1.$$

Corollarium. Transformatio seriei $S=\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+$ etc. etiam hoc modo referri potest:

$$S=\frac{1}{3}+\frac{1}{3}\left(P\left(1+\frac{1}{3^3}\right)\left(1-\frac{1}{5^3}\right)\left(1+\frac{1}{7^3}\right)(\text{etc. }-1+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{11^3}-\text{ etc.})\right)$$
$$+\frac{1}{5}\left(Q\left(1+\frac{1}{3^5}\right)\left(1-\frac{1}{5^5}\right)\left(1+\frac{1}{7^5}\right)(\text{etc. }-1+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{11^5}-\text{ etc.})\right)$$
$$+\frac{1}{7}\left(R\left(1+\frac{1}{3^7}\right)\left(1-\frac{1}{5^7}\right)\left(1+\frac{1}{7^7}\right)(\text{etc. }-1+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{11^7}-\text{ etc.})\right)$$
$$\text{etc.}$$

ubi $P=4Cq^3$, $Q=16Eq^5$, $R=64Gq^7$, etc.

A. m. T. III. p. 104 — 107.

IV. Calculus integralis.

109.

(J. A. Euler.)

Si ponatur $y = \frac{bp + aq}{ap + bq}$, erit $1 - yy = \frac{(aa - bb)(pp - qq)}{(ap + bq)^2}$, ideoque

$$\frac{\sqrt{(aa - bb)(pp - qq)}}{ap + bq} = \sqrt{(1 - yy)} \quad \text{et} \quad dy = \frac{(aa - bb)(pdq - qdp)}{(ap + bq)^2}.$$

Hinc fit $\frac{dy}{\sqrt{(1 - yy)}} = \frac{(pdq - qdp)\sqrt{(aa - bb)}}{(ap + bq)\sqrt{(pp - qq)}}$, ergo

$$\int \frac{(pdq - qdp)\sqrt{(aa - bb)}}{(ap + bq)\sqrt{(pp - qq)}} = \text{Arc. sin} \frac{bp + aq}{ap + bq}.$$

I. Sit $p = 1$ et $q = x$, erit $\int \frac{dx \sqrt{(aa - bb)}}{(a + bx)\sqrt{(1 - xx)}} = \text{Arc. sin} \frac{b + ax}{a + bx}$.

II. Sit $p = 1 + xx$ et $q = x\sqrt{2}$, erit $pdq - qdp = (1 - xx)\,dx\sqrt{2}$ et $pp - qq = 1 + x^4$, unde fiet

$$\int \frac{dx\,(1 - xx)\sqrt{2}\,(aa - bb)}{(a(1 + xx) + bx\sqrt{2})\sqrt{(1 + x^4)}} = \text{Arc. sin} \frac{b(1 + xx) + ax\sqrt{2}}{a(1 + xx) + bx\sqrt{2}}.$$

A. m. T. T. p. 235.

110.

(J. A. Euler.)

Specimen methodi facilis Analysin infinitorum indeterminatam tractandi.

1. Sit propositum problema de inveniendis curvis algebraicis, quae sint rectificabiles.

Sint coordinatae hujusmodi curvarum x et y, quae ergo quantitates algebraicae esse debent, eritque arcus curvae $= \int \sqrt{(dx^2 + dy^2)}$, qui etiam quantitas algebraica esse debet, quae sit $= s$, atque habebitur

$$\sqrt{(dx^2 + dy^2)} = ds \quad \text{sive} \quad dx^2 + dy^2 = ds^2.$$

2. Ponatur ergo $dx = ds \cos\varphi$ et $dy = ds \sin\varphi$, ubi $\sin\varphi$ et $\cos\varphi$ algebraice exprimi debent. Ponatur $x = s\cos\varphi + p$ et $y = s\sin\varphi + q$, atque ut hinc fiat $dx = ds\cos\varphi$ et $dy = ds\sin\varphi$, non obstante variabilitate quantitatum p, q et φ, necesse est ut sit $-sd\varphi\sin\varphi + dp = 0$ et $sd\varphi\cos\varphi + dq = 0$; unde patebit, quomodo hae quantitates a se invicem pendere debent. Habebimus ergo $s = \frac{dp}{d\varphi\sin\varphi} = \frac{-dq}{d\varphi\cos\varphi}$, quae posterior aequalitas praebet $dp\cos\varphi = -dq\sin\varphi$, hincque $\tang\varphi = -\frac{dp}{dq}$.

3. Sumta ergo inter quantitates p et q relatione quacunque algebraica, ita ut sit vel q functio ipsius p vel p functio ipsius q, erit etiam $\frac{dp}{dq}$ quantitas algebraica: capi ergo debet $\tang\varphi = -\frac{dp}{dq}$, unde tam $\sin\varphi$ quam $\cos\varphi$ definitur, tum vero accipi debet $s = \frac{dp}{d\varphi\sin\varphi} = \frac{-dp}{d.\cos\varphi}$, ideoque etiam s erit quantitas algebraica, uti requiritur.

4. Invento autem s habebimus pro curva quaesita $x = s\cos\varphi + p$ et $y = s\sin\varphi + q$. Sicque ex aequatione algebraica quacunque inter p et q pro lubitu assumta semper curva algebraica rectificabilis deduci potest.

(Lexell.)

5. At si curva desideretur algebraica, cujus rectificatio a data quadratura pendeat, solutio ita institui poterit:

Ponatur $dx^2 + dy^2 = ds^2$, ita ut jam s non debeat esse quantitas algebraica, ac statuatur $dy = ndx$, eritque $ds = dx\sqrt{(1+nn)}$, integralia vero statuantur $y = nx + p$ et $s = x\sqrt{(1+nn)} + q$, atque habebimus $xdn + dp = 0$ et $\frac{nxdn}{\sqrt{(1+nn)}} + dq = 0$, unde fit $x = -\frac{dp}{dn} = -\frac{dq\sqrt{(1+nn)}}{ndn}$, hincque porro $\frac{\sqrt{(1+nn)}}{n} = \frac{dp}{dq}$, ergo obtinebimus $n = \frac{dq}{\sqrt{(dp^2 - dq^2)}}$ et $\sqrt{(1+nn)} = \frac{dp}{\sqrt{dp^2 - dq^2)}}$. Quo valore ipsius n invento pro curva quaesita colligimus

$$x = -\frac{dp}{dn}, \quad y = nx + p = -\frac{ndp}{dn} + p = \frac{pdn - ndp}{dn} \quad \text{et} \quad s = -\frac{dp\sqrt{(1+nn)}}{dn} + q.$$

6. Hic ergo q non debet esse quantitas algebraica, sed tamen ejusmodi, ut quantitas $\frac{dp}{dq}$ fiat quantitas algebraica. Ad hoc praestandum sit $\int Pdp$ quadratura illa, a qua rectificatio pendere debet, ita ut summa Pdp non sit quantitas algebraica, ac ponatur $q = \int Pdp$, unde tamen fiet $\frac{dq}{dp} = P$, hinc autem fit $n = \frac{P}{\sqrt{(1-PP)}}$ et $\sqrt{(1+nn)} = \frac{1}{\sqrt{(1-PP)}}$; et nunc curva quaesita his formulis definitur

$$x = -\frac{dp\,(1-PP)^{\frac{3}{2}}}{dP}, \quad y = -\frac{Pdp\,(1-PP)}{dP} + p$$

quae sunt quantitates algebraicae; at vero arcus curvae prodit

$$s = -\frac{dp\,(1-PP)}{dP} + \int Pdp.$$

7. Haec solutio adhuc generalior reddi potest; sumta enim pro T functione quacunque algebraica ipsius p, si capiatur $q = T + \int Pdp$, tum $\frac{dq}{dp}$ ac propterea etiam n fiet quantitas algebraica, ac proinde etiam x et y, at vero pro arcu habebitur $s = -\frac{dp\sqrt{(1+nn)}}{dn} + T + \int Pdp$.

8. At solutio adhuc generalior reddi potest, si pro ν accipiatur functio quaecunque algebraica ipsius p; tum vero V ejusmodi functionem algebraicam ipsius ν denotet, ut $\int Vd\nu$ quadraturam praescriptam involvat; problemati enim satisfiet ponendo $q = T + \int Vd\nu$.

9. Si insuper haec conditio adjiciatur, ut non obstante, quod curva non sit rectificabilis, tamen unum, vel duos, vel tres, vel quotcunque volueris habeat arcus absolute rectificabiles. Hic scilicet totum negotium huc redit, ut in postrema solutione $\int Vd\nu$ certis casibus evanescat, seu exhiberi debet ejusmodi curva algebraica, cujus area in genere sit $\int Vd\nu$, quae tamen certis casibus evanescat.

10. Quadratura proposita est area certae abscissae respondens, ac pro abscissa $= z$ designetur area per $\Pi : z$, ita ut sit $\Pi : z = \int Zdz$ siquidem Z applicatam denotet. Aream autem ita definiri ponamus, ut sit $\Pi : 0 = 0$. Quodsi jam area desideretur, quae casibus $p = \alpha$, $p = \beta$, $p = \gamma$ evanescat, tantum capiatur $Z = (p-\alpha)(p-\beta)(p-\gamma)$, quocirca in solutione superiori postrema sumatur $\nu = (p-\alpha)(p-\beta)(p-\gamma)$ etc. vel generalius

$$\nu = (p-\alpha)(p-\beta)(p-\gamma) \text{ etc. } P;$$

tum enim sumta pro V tali functione ipsius ν, ut proposita quadratura obtineatur, tum curva ibi descripta absolute erit rectificabilis casibus $p-\alpha$, $p-\beta$, $p-\gamma$, etc.

His expositis aggrediamur simili ratione problema nostrum principale, quo debet esse $dx^2 \sin^2 y + dy^2 = dr^2$, ubi litterae x, y et r sunt arcus circulares, quorum sinus cosinusve demum fiunt quantitates algebraicae. Nunc autem analysis nostra ordinem retrogradum teneat. Incipiamus igitur a positione $dx \sin y = dr \sin \omega$ et $dy = dr \cos \omega$; quia autem non y, sed $\sin y$ vel $\cos y$ debet esse quantitas algebraica, posteriorem aequationem ita referamus: $dy \sin y = dr \cos \omega \sin y$, et integrale debet esse algebraicum. Quod ut fieri possit statuamus

$$\cos \omega \sin y = p \cos r + q \sin r,$$

ut sit $$dy \sin y = pdr \cos r + qdr \sin r$$

cujus integrale ponamus $p \sin r - q \cos r = -\cos y$, aut $\cos y = q \cos r - p \sin r$, esseque debeat

$$\sin r \,.\, dp - \cos r \,.\, dq = 0 \quad \text{seu} \quad \tang r = \frac{dq}{dp}.$$

Deinde ob $\cos y$ inventum etiam $\sin y$ innotescit, unde ex facta hypothesi $\cos \omega \sin y = p \cos r + q \sin r$ obtinemus $\cos \omega = \frac{p \cos r + q \sin r}{\sin y}$. (Si haec cum superioribus comparentur, videmus esse $q = \cos \theta$, $p = -\sin \theta \cos \psi$, unde pulchre sequitur

$$\cos \omega = \frac{\cos \theta \sin r - \sin \theta \cos r \cos \psi}{\sin y}$$

deinde etiam eleganter consentit valor

$$\tang r = \frac{dq}{dp} = \frac{d\theta \sin \theta}{d(\sin \theta \cos \psi)}$$

prorsus etiam ut supra.)

Videamus nunc etiam quomodo pro altera parte dx ratiocinium prosequi debeat: Erat autem

$$dx \sin y = dr \sin \omega, \quad \text{unde fit} \quad dx = \frac{dr \sin \omega}{\sin y},$$

at jam invenimus

$$\cos y = q \cos r - p \sin r \quad \text{et} \quad \sin y = \sqrt{(1 + 2pq \sin r \cos r - qq \cos^2 r - pp \sin^2 r)} \quad \text{et} \quad \cos \omega = \frac{p \cos r + q \sin r}{\sin y},$$

unde $\sin \omega = \frac{\sqrt{(1 - pp - qq)}}{\sin y}$ atque $dx = \frac{dr}{\sqrt{(1 - pp - qq)}}$. (Consulamus iterum primam solutionem, ubi erat $dx = d\xi + d\varphi$ eritque $d\varphi = \frac{d\theta}{\tang \psi \sin \theta}$; jam autem invenimus

$$p = -\sin \theta \cos \psi, \ q = \cos \theta, \ \sin \theta = \sqrt{(1 - qq)}, \ d\theta = -\frac{dq}{\sqrt{(1 - qq)}},$$

$$\cos \psi = -\frac{p}{\sqrt{(1 - qq)}}, \ \sin \psi = \frac{\sqrt{(1 - pp - qq)}}{\sqrt{(1 - qq)}}, \ \tang \psi = -\frac{\sqrt{(1 - pp - qq)}}{p},$$

consequenter

$$d\varphi = \frac{pdq}{(1 - qq)\sqrt{(1 - pp - qq)}},$$

quo valore substituto colligitur

$$d\xi = dx - d\varphi = \frac{1}{\sqrt{(1 - pp - qq)}} \left(\frac{pdq}{1 - qq} - \frac{dqddp}{dq^2 + dp^2} \right) = \frac{1}{\sqrt{(1 - pp - qq)}} \left(\frac{pdq^3 + pdp^2 dq - dqddp + qqdq \,.\, ddp}{(1 - qq)(dq^2 + dp^2)} \right)$$

quod novimus esse differentiale arcus ξ cujus cotangens est

$$= \frac{\frac{dp}{dq}(1 - qq) + pq}{\sqrt{(1 - pp - qq)}}.$$

Quarum formularum evolutio nimis est difficilis)

Aliud tentamen. Quum esse debeat $dx = \frac{dr \sin \omega}{\sin y}$, ponamus $dx = d\xi + d\varphi$, eritque

$$d\xi = dx - d\varphi = \frac{dr \sin \omega}{\sin y} - d\varphi,$$

ubi ξ et φ sunt etiam arcus, quare dividatur per $\sin^2 \xi$, ut habeatur

$$\frac{d\xi}{\sin^2 \xi} = \frac{dr \sin \omega}{\sin y \sin^2 \xi} - \frac{d\varphi}{\sin^2 \xi},$$

quod ergo integrabile esse debet. In hunc finem statuatur

$$\frac{\sin\omega}{\sin y\sin^2\xi}=\frac{u}{\sin^2 r},$$

ubi u non involvat r, simulque integrale fingatur $-\cot\xi=-u\cot r+t$; unde differentiando fit

$$\frac{d\xi}{\sin^2\xi}=\frac{udr}{\sin^2 r}-du\cot r+dt=\frac{udr}{\sin r}-\frac{d\varphi}{\sin^2\xi},$$

sicque erit $du\cot r-dt=\frac{d\varphi}{\sin^2\xi}$. Ex quo colligitur $d\varphi=(du\cot r-dt)\sin^2\xi$, quia $\cot\xi=\frac{u\cos r-t\sin r}{\sin r}$, hinc

$$\sin\xi=\frac{\sin r}{\sqrt{(\sin^2 r+(u\cos r-t\sin r)^2)}},\quad \cos\xi=\frac{u\cos r-t\sin r}{\sqrt{(\sin^2 r+(u\cos r-t\sin r)^2)}}.$$

Jam vero posueramus $\frac{\sin\omega}{\sin y\sin^2\xi}=\frac{u}{\sin^2 r}$, ubi loco $\sin\xi$ valorem substituendo

$$\sin\omega\left(uu\cos^2 r-2tu\sin r\cos r+(1+tt)\sin^2 r\right)=u\sin y$$

at vero praecedens operatio praebuerat $\sin\omega=\frac{\sqrt{(1-pp-qq)}}{\sin y}$, qui valor substitutus dat

$$(uu\cos^2 r-2tu\sin r\cos r+(1+tt)\sin^2 r)(1-pp-qq)=u\sin^2 y=u\,(1+2pq\sin r\cos r-pp\sin^2 r-qq\cos^2 r)$$

ex qua aequatione relatio inter p, q et t, u debet definiri, ubi imprimis notasse juvabit, has quantitates p, q, t, u angulum r non involvere debere; unde sequitur aequationem illam aeque subsistere, sive ponatur $r=0$, sive $r=90^0$. At positio $r=0$ dat $uu\sqrt{(1-pp-qq)}=u\,(1-qq)$ et $u=\frac{1-qq}{\sqrt{(1-pp-qq)}}$; altera positio $r=90^0$ praebet $(1+tt)\sqrt{(1-pp-qq)}=u\,(1-pp)$, quae in illam ducta dat

$$(1+tt)(1-pp-qq)=(1-pp)(1-qq)$$

unde

$$t=-\frac{pq}{\sqrt{(1-pp-qq)}}.$$

Sicque t et u definimus per p et q, atque jam omnibus conditionibus problematis est satisfactum, praeterquam quod adhuc valor anguli φ debet determinari. Verum supra invenimus

$$d\varphi=(du\cot r-dt)\sin^2\xi.$$

At cum sit

$$\cot\xi=\frac{u\cos r-t\sin r}{\sin r}=\frac{(1-qq)\cos r-pq\sin r}{\sin r\sqrt{(1-pp-qq)}}$$

$$du=\frac{pdp\,(1-qq)+qdq\,(-1+2pp+qq)}{(1-pp-qq)^{\frac{3}{2}}}$$

$$dt=\frac{-qdp\,(1-qq)-pdq\,(1-pp)}{(1-pp-qq)^{\frac{3}{2}}}$$

praeterea est $\sin r=\frac{dq}{\sqrt{(dp^2+dq^2)}}$ et $\cot r=\frac{dp}{dq}$, erit ergo

$$du\cot r-dt=\frac{pdp^2\,(1-qq)-2qdpdq\,(1-pp-qq)+pdq^2\,(1-pp)}{dq\,(1-pp-qq)^{\frac{3}{2}}}.$$

In superiori tentamine omnia manent usque ad valorem ipsius t, qui cum inventus sit ex aequatione quadratica, sumi debet $t=-\frac{pq}{\sqrt{(1-pp-qq)}}$, unde statim prodit

$$du\cot r-dt=dq\,.\,\frac{(p\cot^2 r\,(1-qq)-q\cot r\,(1-2pp-qq)+q\cot r\,(1-qq)+p\,(1-pp))}{(1-pp-qq)^{\frac{3}{2}}}$$

quae posito $dp=dq\cot r$ abit in

$$du\cot r-dt=pdq\left(\frac{p\cot^2 r\,(1-qq)+2pq\cot r+p\,(1-pp)}{(1-pp-qq)^{\frac{3}{2}}}\right).$$

Cum vero $\cot\xi = u\cot r - t = \frac{(1-qq)\cot r + pq}{\sqrt{(1-pp-qq)}}$, erit

$$\sin^2\xi = \frac{1-pp-qq}{(1-qq)(1-pp+2pq\cot r+(1-qq)\cot^2 r)}$$

unde denique fit

$$d\varphi = \frac{pdq}{(1-qq)\sqrt{(1-pp-qq)}}.$$

Ubi commode usu venit, ut solum differentiale dq hic insit, alterum vero dp una cum tangente r ex calculo excesserit; hanc ob rem non p per q ita definiamus, ut haec formula ad arcum circuli reducatur, sed potius relationem inter quantitates q et $\sin\varphi$ tanquam cognitam spectemus, quam adeo pro lubitu assumere licebit; tum igitur $\frac{d\varphi}{dq}$ erit quantitas algebraica et vocetur $\frac{d\varphi}{dq} = s$, atque ex aequatione $s = \frac{p}{(1-qq)\sqrt{(1-pp-qq)}}$ determinemus quantitatem p; quia enim $\sqrt{(1-pp-qq)} = \frac{p}{s(1-qq)}$ erit $1-pp-qq = \frac{pp}{ss(1-qq)^2}$, unde reperiri potest p et sequens solutio completa concluditur:

1. Constituta relatione quacunque algebraica inter $\sin\varphi$ et q, positoque $\frac{d\varphi}{dq} = s$, quaeratur quantitas p ex hac aequatione $\sqrt{(1-pp-qq)} = \frac{p}{s(1-qq)}$.

2. Inventa hac quantitate p sumatur $\tang r = \frac{dq}{dp}$, atque hinc porro

$$u = \frac{1-qq}{\sqrt{(1-pp-qq)}},\quad t = -\frac{pq}{\sqrt{(1-pp-qq)}} = -sq(1-qq).$$

3. Deinde quaeratur arcus y, ut sit $\cos y = q\cos r - p\sin r$.

4. Hinc porro angulus ξ, ut sit $\cot\xi = u\cot r - t$, quo angulo invento habebimus $x = \xi + \varphi$, sicque problema expedite est solutum.

NB. Hic autem etiamnunc desideratur criterium, ex quo pateat in formula $d\varphi = (du\cot r - dt)\sin^2\xi$ quantitatem $\cot r$ ex calculo tolli; in hoc ipso enim vis methodi consistit, ut r ex calculo excedat, propterea quod $\tang r$ per dp et dq determinatur. Ad hoc ergo criterium, ob

$$\sin^2\xi = \frac{1}{1+tt-2tu\cot r+uu\cot^2 r}$$

requiritur, ut ostendatur ex expressione

$$d\varphi = \frac{du\cot r - dt}{uu\cot^2 r - 2tu\cot r + 1 + tt}$$

omnino tolli $\cot r$, propterea quod sit $\tang r = \frac{dq}{dp}$; hinc enim etiam ratio differentialium dt et du quantitatem $\cot r$ involvet, sed quomodo? Supra invenimus $u\sqrt{(1-pp-qq)} = 1-qq$, $(1+tt)(1-pp-qq) = (1-pp)(1-qq)$, $(1+tt)\sqrt{(1-pp-qq)} = u(1-pp)$, $\frac{1+tt}{u} = \frac{u(1-pp)}{1-qq}$, $\frac{(1+tt)(1-qq)}{uu} = 1-pp$, $pp = 1 - \frac{(1+tt)(1-qq)}{uu}$, $1-pp-qq = \frac{1+tt-qq(1+tt+uu)}{uu}$, $1+tt-qq(1+tt+uu) = 1-2qq+q^4$, unde pro determinando q haberetur haec aequatio

$$q^4 - qq(1-tt-uu) = tt$$

Unde patet praestare loco t et u valores per p et q introducere, uti fecimus, ubi ob $dp = dq\cot r$ statim se prodidit criterium quaesitum.

Notatu etiam dignum est, quod prodeat $d\varphi = -\frac{tdq}{q(1-qq)}$, ubi jam neque p neque u inest, ita ut ex relatione $\frac{d\varphi}{dq} = s$ habeatur $t = -sq(1-qq)$.

A. m. T. I. p. 37 — 41.