SUPPLEMENTUM V.

AD TOM. I. CAP. VIII.

DE

VALORIBUS INTEGRALIUM

QUOS CERTIS TANTUM CASIBUS RECIPIUNT.


§ 1. Cum ehi saepius occurrissent formularum differentiales, quae per logarithmum quantitatis variabilis erant divisae, veluti \( \frac{d z}{z} \), nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda, quin etiam maxime difficile videbatur eorum valores saltem vero proxime assignare. Quod quidem ad formulam integralem simplicissimam hujus generis \( \int \frac{dz}{z} \) attinet, facile patet, si eam ita integrari concipiam, ut evanesca positio \( z = 0 \), tum vero statuatur \( z = 1 \), quantitatem infinite magnam esse proditurum, quod si enim variabilis \( z \) jam proxime ad unitatem accedere vit, ut si \( z = 1 - u \), existente \( u \) quantitate infinite parva, tum ob

\[ \partial z = -\partial u \text{ et } l z = l(1 - u) = -u, \]

haec formula erit \( \int \frac{du}{u} \), cujus valor utique fit infinitus. At vero dan-
tur omnino hujusmodi formularum integrales \( \int \frac{dz}{z} \), quae, etiamsi \( \psi - \)
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natur. \( x = 1 \), tamen valores finitae magnitudinis sortiuntur: quod de-
termínasse eo magis operae pretium videitur, quod nulla adhuc cogn-
nita est via istos valores investigandi.

§. 2. Consideremus exempli gratia hanc formulam salis
simplicem \( f(\frac{x-1}{x}) \frac{3}{x} \), quae memorata lege integrata valorem finitum
habere facile ostendi potest. Posito enim \( \frac{x-1}{x} = y \), ut formula
nostra fiat \( f' \ y \partial z \), ideoque exprimat aream curvae, pro abscissa
\( z \) applicatam habentis \( y \), ista area a termino \( z = 0 \) usque ad
terminum \( z = 1 \) extensa utique valorem finitum non multo majorem
quam \( \frac{1}{2} \) repraesentabit; posita enim abscissa \( z = 0 \), fiat etiam appli-
cata \( y = 0 \), at sumta \( z = 1 \), pro applicata \( y = \frac{x-1}{x} \) tam nume-
rator quam denominator evanescit, ergo eorum loco substitutus suis
differentialibus, fact \( y = z = 1 \). Pro abscissis autem mediis ponamus
\( z = e^{-n} \), existente \( e \) numero, cuius logarithmus hyperbolicus est
unitas, erit

\[
y = \frac{e^{-n} - 1}{-n} = \frac{e^{n} - 1}{n e^{n}}
\]

quae, si \( n \) fuerit numeros valde Magnus, ut abscissa \( z \) fiat minima,
appliçata erit proxime \( y = \frac{1}{2} \); qui ergo valor multo major erit quam
abscissa \( z \); forma scilicet hujus curvae similibus erit figurae adjectae,
ubi \( A \) \( P \) denotat abscissam \( z \) et \( P \) \( M \) applicatam \( y \), abscissae vero
\( A \) \( B \) \( = 1 \) respondet\( 1 \) applicata \( B \) \( C \) \( = 1 \), qua curva descripta, Fig. 1.
ejus area \( A \) \( M \) \( C \) \( B \) non multum superabit aream trianguli \( A \) \( B \) \( C \)
quae est \( \frac{1}{2} \).

§. 3. Nuper autem, in aliis investigationibus occupatus,
praeter expectationem invenit, hanc aream æqualem esse logarithmo
hyperbolicó binárió, ita ut ea per fractiones decimales sit \( \frac{1}{7} \).
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2 = 0,6931471805; sequenti autem ratiocinio hoc sum perductus. Cum revèra sit \( l \ z = \frac{z^0 - 1}{1} \), quia differentiando utrinque prodit \( \frac{\partial z}{z} = \frac{\partial z}{z} \), et sumto \( z = 1 \) utraque expressio evanescit, loco 0 scribo \( \frac{1}{i} \), denotante \( i \) numerum infinitum, eritque \( l \ z = i \ (z^i - 1) \), henceque applicata

\[
y = \frac{z - 1}{i(z^i - 1)} = \frac{1 - z}{i(1 - z^i)}
\]

et formula integralis

\[
\int \frac{(1 - z) \partial z}{i(1 - z^i)}.
\]

Nunc igitur statu \( z^i = x \), ut fiat \( z = x^i \), ubi notetur, pro utroque
integrationis termine \( z = 0 \) et \( z = 1 \) etiam fore \( x = 0 \) et \( x = 1 \);
quia igitur hinc fit \( \partial z = i x^{i-1} \partial x \), formula integralis evadit

\[
\int x^{i-1} \partial x \frac{1 - x^i}{(1 - x)}
\]

quam ergo integrari oportet a termino \( x = 0 \) usque ad terminum \( x = 1 \).

§ 4. Spectemus nunc \( i \) ut numerum valde magnum, et
fracio \( \frac{1 - x^i}{1 - x} \) resolvitur in hanc progressionem geometricam

\[1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots + x^{i-1},\]

cuju singul termini in \( x^{i-1} \partial x \) ducti et integrati praebent hanc
seriem

\[
x^i + \frac{x^{i+1}}{i+1} + \frac{x^{i+2}}{i+2} + \frac{x^{i+3}}{i+3} + \cdots + \frac{x^{2i-1}}{2i-1},
\]
quae utique evanescit facto \( x = 0 \). Nunc igitur sumatur \( x = 1 \), et valor quaeusit nostrae formulee integralis erit

\[
\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \cdots + \frac{1}{2i-1},
\]

ubi quidem littera \( i \) denotat numerum infinite magnum, ita ut numerus horum terminorum sit revera infinitus. Nihilo vero minus, quia singuli termini sunt infinite parvi, haece series summam habebit finitam, quam sequenti modo ad seriem ordinariam reducere licet.

§. 5. Series inventa spectari potest tanquam differentia inter binas sequentes progressiones harmonicas

\[
A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2i-1}
\]

\[
B = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots + \frac{1}{2i-1}
\]

quandoquidem differentia \( A - B \) ipsam seriem inventam exhibet; quia autem numerus terminorum seriei \( A \) est \( 2i - 1 \), seriei vero \( B = i - 1 \), ille duplo major est quam hic, quo circi, ut seriem regularem obtineamus, singulos terminos seriei \( B \) per saltum a seriei \( A \) termino secundo, quarto, sexto, octavo etc. auferamus, quo pacto simul ad finem utiusque pervenietur, eritque

\[
A - B = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots
eq etc.
\]

in infinitum, cujus ergo valor est \( \frac{1}{2} \), ita ut nunc quidem solide sit demonstratum, formulee integralis propositae \( \int \frac{(x^m - 1) \, dx}{lx} \), casu \( x = 1 \), valorem revera esse \( = \frac{1}{2} \).

§. 6. Simile ratiocinium etiam ad formulam integralem generaliorem \( \int \frac{(z^m - 1) \, dz}{lx} \) accommodari potest, ut tandem reperietur, casu \( z = 1 \) ejus valorem fore \( \frac{1}{m+1} \); quia igitur parsi modo erit
\[
\int \frac{(z^n - 1) \partial z}{l z} = l(n + 1),
\]

si hanc ab illa subtrahamus, prodit sequens integratio

\[
\int \frac{(z^m - z^n) \partial z}{l z} = l \frac{m - 1}{n + 1},
\]

si scilicet integratio a termino \( z = 0 \) usque ad terminum \( z = 1 \) extendatur.

§ 7. Quia autem haec demonstratio per quantitates infinitas et infinite parvas procedit, merito aliam methodum planam et consuetam desideramus, quae ad easdem summas perducere valeat; quae quidem investigatio maxime ardua videbitur. Interim tamen, cum nuper consideratio functionum duas variabiles involventium me ad integrationem formularum differentialium prorsus singularium perduxisset, quae alius methodis frustra tentantur, ex eodem principio quoque integrationes hic exhibitas derivandas esse intellexi. Hanc igitur methodum tanquam fontem prorsus novum, ex quo integrationes, alius methodis inaccessas, haurire liceat, clare et perspicue explicabo, cui negotio istam disquisitionem praecipue destinavi.

Lemma I.

§ 8. Si \( P \) fuerit functio quaequeunque duarum variabilium \( z \) et \( u \), ac ponatur \( f P \partial z = S \), ut etiam \( S \) sit functio binarum variabilium \( z \) et \( u \), tum erit

\[
\int \partial z (\frac{\partial P}{\partial u}) = (\frac{\partial S}{\partial u}).
\]

Demonstratio.

Cum in integratione formulae \( f P \partial z \) sola \( z \) ut variabilis spectetur, erit \( (\frac{\partial S}{\partial z}) = P \), quae formula demueo differentiata, sola \( u \)
pro variabili habita, praecebit \( \frac{\partial^2 S}{\partial u \partial z} = \frac{\partial P}{\partial u} \), quae in \( \partial z \) ducta et
integrata product \( \frac{\partial S}{\partial u} = \int \partial z \left( \frac{\partial P}{\partial u} \right) \), quandoquidem ex principii
calculi integralis est
\[ \int \partial z \left( \frac{\partial S}{\partial u} \right) = \frac{\partial S}{\partial u} \] q. e. d.

**Corollarium I.**

§ 9. Eodem modo per hujusmodi differentialia, ubi tantum \( u \) pro variabili spectatur, ulterior progredi licet, unde sequentes orientur integrationes
\[ \left( \frac{\partial^2 S}{\partial u^2} \right) = \int \partial z \left( \frac{\partial P}{\partial u^2} \right) \] et
\[ \left( \frac{\partial^3 S}{\partial u^3} \right) = \int \partial z \left( \frac{\partial^3 P}{\partial u^3} \right) \]
estc. etc.

**Corollarium 2.**

§ 10. Quod si ergo formula \( \int P \partial z \) fuerit integrabilis, ita ut ejus integrale \( S \) exhiberi possit, tum etiam omnes istae formuleae integrales
\[ \int \partial z \left( \frac{\partial P}{\partial u} \right), \int \partial z \left( \frac{\partial^2 P}{\partial u^2} \right), \int \partial z \left( \frac{\partial^3 P}{\partial u^3} \right) \] etc.
integrationem admittent, atque adeo ipsa integralia exhiberi poterunt.

**Scholion.**

§ 11. Ex his quidem formulis si in genere tractentur, parum utilitatis in calculo integrale redudat. At si functio \( P \) ita fuerit comparata, ut integrale \( \int P \partial z \) casu saltem particulari, quo post integrationem variabili \( z \) certam quidam valor putat \( z = a \) tribuitur, commode exhiberi potest, ut hoc casu quantitas \( S \) abeat in functionem solius variabili \( u \) satis simplicem, tum integrationes memoratae perinde locum habebunt, si quidem post singulas integratio-

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nes ponatur $z = a$, atque hinc ad ejusmodi integrationes plerumque pervenitur, quas alis methodis vix, ac ne vix quidem perficere liceat: atque hinc oritur

Primum principium integrationum.

§ 12. Si $P$ ejusmodi fuerit functio binarum variabilium $z$ et $u$, ut valor integralis $\int P \partial z$ saltem cau certo $z = a$ comode exprimi queat, qui valor sit $S$, functio scilicet ipsius $u$ tantum; tum etiam sequentia integralia, si quidem post integrationem pariter statuat $z = a$, commode exhiberimi poterunt, scilicet

$$\int P \partial z = S$$
$$\int \partial z (\frac{\partial P}{\partial u}) = (\frac{\partial S}{\partial u})$$
$$\int \partial z (\frac{\partial^2 P}{\partial u^2}) = (\frac{\partial^2 S}{\partial u^2})$$
$$\int \partial z (\frac{\partial^3 P}{\partial u^3}) = (\frac{\partial^3 S}{\partial u^3})$$
$$\int \partial z (\frac{\partial^4 P}{\partial u^4}) = (\frac{\partial^4 S}{\partial u^4})$$

etc. etc.

Exemplum I.

§ 13. Si fuerit $P = z^u$, erit quidem in genere

$$\int P \partial z = \frac{z^{u+1}}{u+1};$$

unde casu $z = 1$ hic valor satis simplex nascitur $\frac{1}{u+1}$; ita ut sit $S = \frac{1}{u+1}$; cum deinde per differentiationes continuas, dum sola $u$ pro variabili habetur, prodat $\frac{\partial P}{\partial u} = z^u l z$, tum vero $\frac{\partial^2 P}{\partial u^2} = z^u (l z)^2$, porro

$$(\frac{\partial P}{\partial u}) = z^u (l z)_3, \quad (\frac{\partial^2 P}{\partial u^2}) = z^u (l z)^4, \quad \text{etc.}$$
hinc sequentes obtinentur valores integrales, si quidem post singulas integrationes statuatur $z = 1$

\[
\begin{align*}
\int z^u \, dz &= 1 + \frac{1}{u+1} \\
\int z^u \, dz \, (l \, z)^2 &= 1 + \frac{1}{(u+1)^2} \\
\int z^u \, dz \, (l \, z)^3 &= 1 + \frac{1}{(u+1)^3} \\
\int z^u \, dz \, (l \, z)^4 &= 1 + \frac{1}{(u+1)^4} \\
\int z^u \, dz \, (l \, z)^5 &= 1 + \frac{1}{(u+1)^5} \\
\int z^u \, dz \, (l \, z)^6 &= 1 + \frac{1}{(u+1)^6} \\
\int z^u \, dz \, (l \, z)^7 &= 1 + \frac{1}{(u+1)^7} \\
\end{align*}
\]

unde concludimus generaliter fore

\[
\int z^u \, dz \, (l \, z)^n = 1 + \frac{1}{(u+1)^n+1},
\]

ubi signum $-$ valet si $n$ sit numerus par; alterum vero $-$ si $n$ sit numerus impar. Hae quidem integrationes jam aliunde satis sunt notae, id quod mirum non est, quoniam tam simplicem formulam pro $P$ assumimus: breviter igitur repetamus eos casus, quos jam nuper expedivi.

**Exemplum 2.**

§ 14. Si fuerit

\[
P = \frac{z^n - u^{-1} + z^n + u^{-1}}{1 + z^n},
\]

jam dudum demonstravi, formulae $\int P \, dz$ valorum integralem casu quo post integrationem ponitur $z = 1$, esse

\[
S = \frac{\pi}{2 n \cos \frac{\pi u}{2}}.
\]

Hinc ergo cum sit

\[
(P \frac{\partial}{\partial u}) = \frac{z^n - u^{-1} + z^n + u^{-1}}{1 + z^n} \, l \, z,
\]

tum vero
\[
\begin{align*}
\left(\frac{\partial^2 p}{\partial u^2}\right) &= \frac{z^n - u - i + z^n + u - i}{1 + z^{2n}} (l z)^2 \quad \text{et} \\
\left(\frac{\partial^3 p}{\partial u^3}\right) &= -\frac{z^n - u - i + z^n + u - i}{1 + z^{2n}} (l z)^3 \\
\text{etc.} & \quad \text{etc.}
\end{align*}
\]

ex cognito valore \( S \) sequentes nacti sumus integrationes

I. \[
\int \frac{z^n - u - i + z^n + u - i}{1 + z^{2n}} \partial z = S = \frac{\pi}{2n \cos \frac{\pi u}{2n}}
\]

II. \[
\int \frac{z^n - u - i + z^n + u - i}{1 + z^{2n}} \partial z (l z) = \left(\frac{\partial s}{\partial u}\right)
\]

III. \[
\int \frac{z^n - u - i + z^n + u - i}{1 + z^{2n}} \partial z (l z)^2 = \left(\frac{\partial^2 s}{\partial u^2}\right)
\]

IV. \[
\int \frac{z^n - u - i + z^n + u - i}{1 + z^{2n}} \partial z (l z)^3 = \left(\frac{\partial^3 s}{\partial u^3}\right)
\]

V. \[
\int \frac{z^n - u - i + z^n + u - i}{1 + z^{2n}} \partial z (l z)^4 = \left(\frac{\partial^4 s}{\partial u^4}\right)
\]

etc. etc.

\[\text{Exemplum 3.}\]

\[\text{§. 15. Si fuerit} \quad P = \frac{z^n - u - i + z^n + u - i}{1 + z^{2n}}\]

simili modo demonstravi, valorem formulæ integralis \( \int P \partial z \), casu quo post integrationem ponitur \( z = 1 \), fore

\[S = \frac{\pi}{2^n} \tan \frac{\pi u}{2n}\]

atque hinc sequentes integrationes pro eodem casu \( z = 1 \) fuerunt deductae
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\[ \int \frac{\pi^n - u - 1}{1 - z^n} \frac{\pi^n + u - 1}{z^n} \partial z \equiv S = \frac{\pi}{2n} \tan \frac{\pi u}{2n} \]

II. \[ \int \frac{\pi^n - u - 1}{1 - z^n} \frac{\pi^n + u - 1}{z^n} \partial z \equiv (\frac{\partial S}{\partial z}) \]

III. \[ \int \frac{\pi^n - u - 1}{1 - z^n} \frac{\pi^n + u - 1}{z^n} \partial (l_2) \equiv (\frac{\partial S}{\partial u}) \]

IV. \[ \int \frac{\pi^n - u - 1}{1 - z^n} \frac{\pi^n + u - 1}{z^n} \partial (l_3) \equiv (\frac{\partial^2 S}{\partial u^2}) \]

V. \[ \int \frac{\pi^n - u - 1}{1 - z^n} \frac{\pi^n + u - 1}{z^n} \partial (l_4) \equiv (\frac{\partial^3 S}{\partial u^3}) \]

etc.

S c h o l i o n.

§ 16. Quo igitur uberior es fructus ex hoc principio exspectare queamus, praeceptum negotium huc redit, ut ejusmodi functiones binarum variabilium \( z \) et \( u \) pro \( P \) investigemus, ita ut valor formulae integralis saltem certo quodam casu puta \( z = 1 \) succincte assignari possit, quemadmodum in allatis exemplis fieri licuit. Quemadmodum autem hoc principium ex continua differentiae est deductum, ita cedem modo continua integratio ad usum nostrum accommodari poterit.

L e m m a II.

§ 17. Si \( P \) fuerit functio duarum variabilium \( z \) et \( u \), aponatur \( \int P \partial z = S \), ut etiam \( S \) sit functio duarum variabilium \( z \) et \( u \), tum crit \( \int \partial S \partial u = \int \partial z \int P \partial u \), ubi in integralibus formulis \( \int P \partial u \) et \( \int S \partial u \) sola \( u \) pro variabili habetur, in formula autem \( \int \partial z \int P \partial u \) sola \( z \).
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Demonstratio.

Ponatur \( \int S \partial u = V \), ut sit \( S = \left( \frac{\partial V}{\partial u} \right) \), ideoque
\( \left( \frac{\partial V}{\partial u} \right) = \int P \partial z \), eritque \( \left( \frac{\partial \partial V}{\partial z \partial u} \right) = P \);
unde, per \( \partial u \) multiplicando et integrando, erit \( \int P \partial u \), ex quo sequitur
\[ V = \int \partial z \int P \partial u = \int S \partial u \quad q. \ e. \ d. \]

Corollarium 1.

§. 18. Hoc modo etiam integratio repeti potest, unde orientur talis aequatio
\[ \int \partial u / \int S \partial u = \int \partial z \int \partial u \int P \partial u ; \]
hinc autem plerumque parum utilitatis exspectari potest, nisi forte istae integrationes commode succedant.

Corollarium 2.

§. 19. Quod si ergo formula \( \int P \partial z \) fuerit integrabilis,
scilicet \( = S \), altera hinc deducta \( \int \partial z \int P \partial u \) eatenus tantum integrari poterit, quatenus integrale \( \int S \partial u \) integrare licet.

Secundum principium integrationum.

§. 20. Si \( P \) ejusmodi fuerit functio duarum variabilium \( z \)
et \( u \), ut formulae integralis \( \int P \partial z \) valor certo saltem casu, puta \( z = a \), commode exhiberi queat, ita ut hoc casu quantitas \( S \) fiat functio solius variabilis \( u \); tum etiam pro eodem casu \( z = a \) hujus formulae integralis \( \int \partial z \int P \partial u \) valor assignari poterit, si modo formulam \( \int S \partial u \) integrare licerit.
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Exemplum I.

§. 24. Sumamus \( P = z^n \), eritque \( \int P \partial z = \frac{z^{n+1}}{n+1} \); quae formula casu \( z = 1 \) abit in \( \frac{1}{u + 1} \), quod ergo loco \( S \) scribatur. Tum vero quia est
\[
\int P \partial u = \int z^n \partial u = \frac{z^n}{l_z},
\]
et quia
\[
\int S \partial u = l(u+1),
\]
et
\[
\int \frac{z^n \partial z}{l_z} = l(u+1);
\]
si quidem post illam integrationem ponatur \( z = 1 \). Quia autem omnis integratio additionem constantis postulat, hic potius statui oportebat
\[
\int \frac{z^n \partial z}{l_z} = l(u+1) + C;
\]
atque hic quidem facile intelligitur, hanc constantem \( C \) esse debere infinitam, quoniam in formula integrali fractio \( \frac{z^n}{l_z} \) posito \( z = 1 \) fit infinita, ita ut hinc parum pro instituto nostro sequi videatur.

Corollarium 1.

§. 22. Quoniam autem haec constans \( C \) non a variabili \( u \) pendet, ea retinebit eundem valorem, quicumque numeri determinati pro \( u \) accipientur. Sumamus igitur primo \( u = n \), tum vero etiam \( u = n \), ut habeamus istos valores
\[
I. \int \frac{z^n \partial z}{l_z} = l(n+1) + C \]
et
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II. \( \int \frac{Z^n \partial Z}{l \ z} = l(n + 1) + C \),

quarum altera ab altera subtracta relinquet istam integrationem notatuu dignissimam

\[ \int (\frac{Z^n}{l \ z} - \frac{Z^n}{(l \ z)^2}) \partial Z = l^{m+1} \frac{1}{n+1}, \]

quemadmodum jam supra ex longe alius principiis demonstravimus.

Corollarium 2.

§ 23. Si ad alteram integrationem ascendamus, quia

est \( \int P \partial u = \frac{Z^u}{l \ z} \), erit \( \int \partial u \int P \partial u = \frac{Z^u}{(l \ z)^2} \); tum vero ob

\[ \int S \partial u = l(u + 1) + C, \]

\[ \int \partial u \int S \partial u = (u + 1) \left[ l(u + 1) - 1 \right] + C u + D, \]

sicque habebimus

\[ \int \frac{Z^u \partial Z}{(l \ z)^2} = (u + 1) \left[ l(u + 1) - 1 \right] + C u + D, \]

ubi constantes C et D non ab u pendent: quare ut eam eliminemus tres casus determinatos evolvamus

I. \( \int \frac{Z^m \partial Z}{(l \ z)^2} = (m + 1) \left[ l(m + 1) - m + 1 \right] + C m + D, \)

II. \( \int \frac{Z^n \partial Z}{(l \ z)^2} = (n + 1) \left[ l(n + 1) - n + 1 \right] + C n + D, \)

III. \( \int \frac{Z^k \partial Z}{(l \ z)^2} = (k + 1) \left[ l(k + 1) - k + 1 \right] + C k + D, \)

eritque

I - III = (m + 1) \left[ l(m + 1) - (k + 1) \right] + C(m - k) \)

II - III = (n + 1) \left[ l(n + 1) - (k + 1) \right] + k - n + C(n - k) \)
hincque deducimus.

\[
\begin{align*}
(I-III)(n-k)-(II-III)(m-k) = & - \left(\frac{n+1}{(n+k)}(n-k)(m+1)\right) \\
+ & \left(\frac{n+1}{(n+k)}(m-k)(m+1)\right) \\
+ & \left(\frac{n+1}{(n+k)}(n+1)(n-k)\right) \\
- & \left(\frac{n+1}{(n+k)}(m-k)(m+1)\right) \\
+ & \left(\frac{n+1}{(n+k)}(m-k)(n+1)\right) \\
- & \left(\frac{n+1}{(n+k)}(m-k)(m+1)\right)
\end{align*}
\]

atque hinc pervenimus ad sequentem integrationem

\[
\int \frac{\partial z}{(n-k)z^n - (m-k)z^m + (m-n)z^k} = \\
\left(\frac{1}{z}\right)^2 + \left(\frac{m-1}{n-k}\right)(m+1) \\
- \left(\frac{n+1}{m-k}\right)(m+1) \\
+ \left(\frac{n+1}{m-k}\right)(n+1) \\
+ \left(\frac{n+1}{m-k}\right)(m-n)(k+1).
\]

Corollarium 3.

§ 24. Operae pretium erit aliquot casus evolvere, ubi quidem numeros \( m, n \) et \( k \) inter se inaequales accipi convenit, quia aliter omnes termini se destruerent.

I. Sit igitur \( m = 2, n = 1 \) et \( k = 0 \), erit

\[
\int \frac{(z-1)^2dz}{(iz^2)^2} = \frac{3l3 - 4l2}{l_{15}}.
\]

II. Sit \( m = 3, n = 1 \) et \( k = 0 \), eritque

\[
\int \frac{(z^2 - 3z + 2)dz}{(iz^2)^2} = \int \frac{dz(z-1)^2(z+2)}{(iz^2)^2} = \frac{4l4 - 6l2 = 2l2 = l4}{l_{40}}.
\]

III. Sit \( m = 3, n = 2 \) et \( k = 0 \), et erit

\[
\int \frac{(2z^2 - 3z + 1)dz}{(iz^2)^2} = \int \frac{dz(z-1)^2(z+1)}{(iz^2)^2} = \frac{8l4 - 9l3}{l_{50}}.
\]

IV. Sit \( m = 3, n = 2 \) et \( k = 1 \), et prodit

\[
\int \frac{(z^2 - 2z + 2)dz}{(iz^2)^2} = \int \frac{dz(z-1)^2}{(iz^2)^2} = \frac{4l4 - 6l3 = 2l2}{l_{90}}.
\]

Corollarium 4.

§ 25. In his casibus notatu dignum occurrit, quod numeratorem in formulis integralibus factorem habet \((z-t)^2\), quod

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ideo necessario usu venit, ne valores integralium evadant infiniti. Quia enim denominator \((lz)^2\) evanescit casu \(z = l\), si ponamus \(z = 1 - \omega\), existente \(\omega\) infinite parvo, erit
\[
lz = -\omega et (lz)^2 = 1 + \omega \omega.
\]
Necessae ergo est ut in numeratore adsit factor, qui casu \(z = 1 - \omega\) itidem praebet \(\omega \omega\), quod evenit si ibi factor fuerit \((z - 1)^2\).

Scholion.

§ 26. Integratio, quam in corollario primo sumus nacti, ideo omni digna videtur attentione, quod valores integrales inde nati casu \(z = 1\) nullo adhuc modo assignare potuerim, etiamsi tam simpliciter per logarithmos exprimantur. At vero integrationes in corollario secundo inventae, etiamsi multo magis arduae, videantur, tamen ex prioribus ope reductionum cognitarum non difficulter derivari possunt; id quod pro unico casu ostendisse sufficit. Ponamus 

\[
\int \frac{3z(z-1)^2}{(lz)^2} = \frac{p}{lz} + \int \frac{2 \frac{dz}{lz}}{(lz)^2}
\]
erique differentiando 
\[
\frac{\partial z(z-1)^2}{(lz)^2} = \frac{\partial p}{lz} - \frac{p \partial z}{z(lz)^2} + \frac{2 \partial z}{lz},
\]
unde aequatis terminis seorsim vel per \((lz)^2\) vel per \(lz\) divisis, habebimus duas aequalitates
\[
(z - 1)^2 = \frac{p}{z} et \partial p = -g \partial z,
\]
ex quorum priore oritur \(p = -z(z - 1)^2\), hincque 
\[
\frac{\partial p}{\partial z} = 3zz + 4z - 1,
\]
ideoque 
\[
q = 3zz - 4z + 1,
\]
itae ut sit 
\[
\int \frac{3z(z-1)^2}{(lz)^2} = -\frac{z(z-1)^2}{lz} + \int \frac{(3zz - 4z + 1) \frac{dz}{lz}},
\]
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hic autem prius membum posito \( z = 1 \) sponte evanescit; posito enim \( z = 1 - \omega \), ut sit \( i z = -\omega \), et

\[
p = -\omega \omega (1 - \omega),
\]
ideoque

\[
\frac{p}{iz} = \omega (1 - \omega) = 0, \text{ ob } \omega = 0;
\]
posterius vero membum in has partes discerpi potest

\[
3 \int \frac{z - \bar{z}}{iz} \frac{dz}{iz} = \int \frac{z - \bar{z}}{iz} dz
\]

cujus prioris partis integrale est \( 3 l^2 \), posterioris vero \(-1 l 2\);
sieque totum hoc integrale erit

\[
3 l^2 - 1 2 = 3 l 3 - 4 l 2 = l^2
\]

prorsus uti invenimus. Hoc igitur modo si in genere statuimus

\[
\int \frac{\bar{z} z^2}{(iz)^3} = \int \frac{\bar{z} z^2}{iz} + \int \frac{\bar{z} z^2}{iz}
\]

erit differentiando

\[
\frac{\bar{z} z^2}{(iz)^3} = \frac{\partial p}{iz} - \frac{\bar{z} z^2}{iz} + \frac{\bar{z} z^2}{iz},
\]

unde istae duae fluunt aequalitates

\[
p = \bar{z} z \text{ et } q = -\frac{\partial p}{\partial z}.
\]

Jam ut terminus \( \frac{p}{iz} \) evanescat posito \( z = 1 \), numerator \( p \) factorem habere debet \((z - 1)^2\); qui ergo etiam factor esse debet quantitatis \( V \). Sit igitur

\[
V = \frac{U(z - 1)^2}{z}, \text{ etrique } p = -U(z - 1)^2,
\]

unde fit

\[
\partial p = -U(z - 1)^2 - 2U\partial z(z - 1) = (z - 1)[\partial(U(z - 1)) - 2U\partial z],
\]

hincque

\[
q \partial z = (z - 1) [2U\partial z - \partial U(z - 1)];
\]
quia ergo \( q \) factorem habet \( z = 1 \), formula \( \int \frac{q^2 z}{iz} \) semper in partes resolvi potest, quorum integralia per corollarium primum assign.

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nare licet, si modo U fuerit aggregatum ex quotcunque potestatisibus ipsius z; unde sequens deducitur theorema.

Theorema.

§ 27. Si fuerit
\[ P = A z^\alpha + B z^\beta + C z^\gamma + D z^\delta + \text{ etc.} \]
ita ut summa coëfficientium
\[ A + B + C + D + \text{ etc.} = 0, \]
tum erit
\[ \int \frac{\partial z}{l z} = A l(\alpha + 1) + B l(\beta + 1) + C l(\gamma + 1) + D l(\delta + 1) + \text{ etc.} \]
si quidem post integrationem statuatur \( z = 1 \).

Demonstratio.

Cum hoc ipso casu, quo post integrationem ponitur \( z = 1 \),
sit
\[ \int \frac{z^n \partial z}{l z} = l(n + 1) + \Delta, \]
denotante \( \Delta \) illam constantem infinitam integratione ingressam, erit
\[ A \int \frac{z^a \partial z}{l z} = A l(a + 1) + A \Delta, \]
ecdemque modo
\[ B \int \frac{z^b \partial z}{l z} = B l(b + 1) + B \Delta, \]
ec.

si nunc haec integralia omnia in unam summam colligantur, erit ob
\[ (A + B + C + D + \text{ etc.}) \Delta = 0. \]
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integræ quæsitum
\[ \int \frac{\beta \gamma}{iz} = Al(\alpha + 1) + Bl(\beta + 1) + Cl(\gamma + 1) + Dl(\delta + 1) \text{ etc.} \]
\[ q. e. d. \]

**Corollarium 1.**

\[ \S. 28. \text{ Quia supponimus} \]
\[ A + B + C + D + \text{ etc. = 0,} \]
\[ \text{evidens est, formulam} \]
\[ P = A z^\alpha + B z^\beta + C z^\gamma + D z^\delta + \text{ etc.} \]
\[ \text{factorem habere} z - 1, \text{ quamadmodum jam ante notavimus.} \]

**Corollarium 2.**

\[ \S. 29. \text{ Quia est} \]
\[ (z - 1)^n = z^n - \frac{n(n-1)}{1 \cdot 2} z^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} z^{n-3} + \text{ etc.} \]
\[ \text{hoc valore loco} P \text{ posito, erit} A = 1 \text{ et} \alpha = n, \text{ deinde} \]
\[ B = -\frac{n}{1} \text{ et} \beta = n - 1. \]
\[ \text{porro} \]
\[ C = \frac{n(n-1)}{1 \cdot 2} \text{ et} \gamma = n - 2, \text{ etc.} \]
\[ \text{hinc igitur erit} \]
\[ \int \frac{(z - 1)^n}{iz} \, dz = l(n + 1) - \frac{n(n-1)}{1 \cdot 2} l(n-1) - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} l(n-2) + \text{ etc.} \]
\[ + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} l(n - 3) + \text{ etc.} \]
\[ \text{si modo exponens} n \text{ fuerit nihil major, vel saltam unitate non minor, quia alioquin casu} z = 1 \text{ fractio} \frac{(z - 1)^n}{iz} \text{ fiet infinita;} \]
\[ \text{hoc autem non obstante area supra considerata fiat finita, ita ut} \]
\[ \text{sufficiat, dummodo sit} n \geq 0. \]
§ 30. Sit
\[ \frac{x^n - x^{-1} + x^{n+1} - x^{-1}}{1 + x^{2n}} \],
\[ \int P \partial z = \frac{\pi}{2n \cos \frac{\pi n}{2n}} \]
si quidem post integrationem ponatur \( z = 1 \), quem ergo valorem litterae \( S \) tribuimus. Nunc spectata \( z = 1 \) ut constante, erit
\[ \int P \partial u = \frac{1}{1 + x^{2n}} \left( \int x^n - x^{-1} \partial u + \int x^{n+1} - x^{-1} \partial u \right), \]
ideoque
\[ \int P \partial u = \frac{x^n - x^{-1} + x^n + x^{-1}}{(1 + x^{2n})lx} \],
unde fiet
\[ \int S \partial u = \int \frac{x^n - x^{-1} + x^n + x^{-1}}{1 + x^{2n}} \cdot \frac{\partial z}{lx} \],
cum igitur sit \( \cos \frac{\pi n}{2n} = \sin \frac{\pi (n-u)}{2n} \), erit
\[ \int S \partial u = \int \frac{\pi \partial u}{2n \sin \frac{\pi (n-u)}{2n}} \],
hinc si ponamus
\[ \frac{\pi (n-u)}{2n} = \Phi \],
\[ \partial \Phi = - \frac{\pi \partial u}{2n} \],
ideoque
\[ \int S \partial u = - \int \frac{\partial \Phi}{\sin \Phi} = - l \tan \frac{1}{2} \Phi \],
quocirca habeimus
\[ \int S \partial u = - l \tan \frac{\pi (n-u)}{4n} \].
Ita ut positio post integrationem \( z = 1 \), assecuti sumus hanc integrationem
\[ \int \frac{x^n - x^{-1} + x^n + x^{-1}}{1 + x^{2n}} \cdot \frac{\partial z}{lx} = - l \tan \frac{\pi (n-u)}{4n} \]
\[ = - l \tan \frac{\pi (n+u)}{4n} \].
Exemplum 3.

§ 31. Sit

\[ P = \frac{z^{n-u-1} - z^{n-u-1}}{1 - z^{2n}}, \]

erit

\[ \int P \partial z = \frac{\pi}{2n} \tan \frac{\pi u}{2n} = S. \]

unde fit

\[ \int S \partial u = -l \cos \frac{\pi u}{2n}, \]

hinc cum sit

\[ \int P \partial u = -\frac{z^{n-u-1} - z^{n-u-1}}{(1 - z^{2n}) l z}, \]

nanciscimur sequentem integrationem, si quidem integrale a termino

\[ z = 0 \]

usque ad terminum \( z = 1 \) extendatur,

\[ \int \frac{z^{n-u-1} + z^{n-u-1}}{1 - z^{2n}} \cdot \frac{\partial z}{l z} = -l \cos \frac{\pi u}{2n}. \]

Haece quidem duo posteriora exempla jam ante fusius expedivi;

unde iis magis evolvendis non immoror, sed ad sequens problema

progressor.

Problema.

§ 32. Si proponantur hae duae series infinitae

\[ P = z \cos u + z^2 \cos 2u + z^3 \cos 3u + \ldots + \text{etc. et} \]

\[ Q = z \sin u + z^2 \sin 2u + z^3 \sin 3u + \ldots + \text{etc.} \]

quae binas variabiles \( z \) et \( u \) involvunt, invenire relationes inter

formulas integrales \( \int \frac{P \partial z}{z}, \int P \partial u \) et \( \int \frac{Q \partial z}{z}, \int Q \partial u \), aliasque for-

mulas integrales per continuam integrationem inde natas.
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Solutio.

Cum utraque series sit recurrens, reperitur per formulas finitas

\[ P = \frac{z \cos u - z u}{1 - 2 z \cos u + z^2} \quad \text{et} \quad Q = \frac{z \sin u}{1 - 2 z \cos u + z^2}, \]

unde fit

\[ \int \frac{P \, \partial z}{z} = \int \frac{\partial \left(\frac{z \cos u - z u}{1 - 2 z \cos u + z^2}\right)}{z} = - \frac{1}{2} \sqrt{1 - 2 z \cos u + z^2} \] \( \text{et} \)

\[ \int Q \, \partial u = \int \frac{z \sin u}{1 - 2 z \cos u + z^2} = - \frac{1}{2} \sqrt{1 - 2 z \cos u + z^2}, \]

ita ut sit

\[ \int \frac{P \, \partial z}{z} = - \int Q \, \partial u; \]

tum vero etiam edit

\[ \int \frac{Q \, \partial z}{z} = \int \frac{\partial \left(\frac{z \sin u}{1 - 2 z \cos u + z^2}\right)}{z} = \text{arc. tang.} \frac{z \sin u}{1 - 2 z \cos u + z^2}, \]

at si iste arcus differentiaetur sumto solo angulo \( u \) variabili, erit

\[ \frac{1}{\partial u} \partial \, \text{arc. tang.} \frac{z \sin u}{1 - 2 z \cos u + z^2} = \frac{z \cos u - z u}{1 - 2 z \cos u + z^2}, \]

ita ut sit

\[ \int \frac{Q \, \partial z}{z} = \int P \, \partial u \]

§. 33. Verum eadem relationes facilius ex ipsis seriebus derivantur: cum enim sit

\[ P = z \cos u + z^2 \cos 2 u + z^3 \cos 3 u + z^4 \cos 4 u + \cdots \]

erit

\[ \int \frac{P \, \partial z}{z} = \frac{z \cos u}{1} + \frac{z \cos 2 u}{2} + \frac{z^3 \cos 3 u}{3} + \cdots \]

\[ \int P \, \partial u = \frac{z \sin u}{1} + \frac{z \sin 2 u}{2} + \frac{z^3 \sin 3 u}{3} + \cdots \]

et quia est

\[ Q = z \sin u + 2 z \sin 2 u + 3 z^3 \sin 3 u + \cdots \]

erit

\[ \int \frac{Q \, \partial z}{z} = \frac{z \sin u}{1} + \frac{z \sin 2 u}{2} + \frac{z^3 \sin 3 u}{3} + \cdots \]

\[ \int Q \, \partial u = \frac{z \cos u}{1} - \frac{z \cos 2 u}{2} + \frac{z^3 \cos 3 u}{3} + \cdots \]
unde manifestum est fore
\[
\int \frac{\partial^3 P}{\partial z} = -\int Q \partial u \quad \text{et} \quad \int \frac{\partial^3 Q}{\partial z} = -\int P \partial u.
\]

§. 34. Quo hoc modo ulterior progressi liceat, statuamus brevitatis gratia
\[
P' = \frac{x \cos u}{1} + \frac{x \cos 2u}{2} + \frac{x^2 \cos 3u}{3} + \text{etc. et } Q' = \frac{x \sin u}{1} + \frac{x \sin 2u}{2} + \frac{x^2 \sin 3u}{3} + \text{etc.}
\]
\[
P'' = \frac{x \cos u}{1^2} + \frac{x \cos 2u}{2^2} + \frac{x^2 \cos 3u}{3^2} + \text{etc. et } Q'' = \frac{x \sin u}{1^2} + \frac{x \sin 2u}{2^2} + \frac{x^2 \sin 3u}{3^2} + \text{etc.}
\]
\[
P''' = \frac{x \cos u}{1^3} + \frac{x \cos 2u}{2^3} + \frac{x^2 \cos 3u}{3^3} + \text{etc. et } Q''' = \frac{x \sin u}{1^3} + \frac{x \sin 2u}{2^3} + \frac{x^2 \sin 3u}{3^3} + \text{etc.}
\]
\[
P'''' = \frac{x \cos u}{1^4} + \frac{x \cos 2u}{2^4} + \frac{x^2 \cos 3u}{3^4} + \text{etc. et } Q'''' = \frac{x \sin u}{1^4} + \frac{x \sin 2u}{2^4} + \frac{x^2 \sin 3u}{3^4} + \text{etc.}
\]

et hinc comparationes ante inventae continuabuntur
\[
P' = \int_{x} P' \partial z = -\int Q \partial u, \quad Q' = \int_{x} Q' \partial z = \int P \partial u,
\]
\[
P'' = \int_{x} P'' \partial z = -\int Q' \partial u, \quad Q'' = \int_{x} Q'' \partial z = \int P' \partial u,
\]
\[
P''' = \int_{x} P''' \partial z = -\int Q'' \partial u, \quad Q''' = \int_{x} Q''' \partial z = \int P'' \partial u,
\]
\[
P'''' = \int_{x} P'''' \partial z = -\int Q''' \partial u, \quad Q'''' = \int_{x} Q'''' \partial z = \int P''' \partial u.
\]

unde plurae insignes relationes deduci possunt.

§. 35. Maxime autem notatu dignae et ad nostrum institutum accommodatae sunt eae relationes, ubi formularum integrales in quibus sola \(z\) est variabilis, reducuntur ad alias formularum integrales, in quibus sola \(u\) est variabilis; quae sequuntur
\[
P' = \int_{x} P' \partial z = -\int Q \partial u,
\]
\[
P'' = \int_{x} P'' \partial z = -\int Q' \partial u \quad \text{et} \quad P' \partial u \partial Q,
\]
\[
P''' = \int_{x} P''' \partial z = -\int Q'' \partial u \quad \text{et} \quad P' \partial u \partial Q,
\]

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\[ P''' = \frac{\partial^3 P}{\partial x^3} \]
\[ Q'' = \frac{\partial^2 Q}{\partial x^2} \]
\[ \text{etc.} \]
\[ Q' = \frac{\partial Q}{\partial x} \]
\[ \text{etc.} \]

\[ Q' = \int \frac{\partial Q}{\partial x} \ dx = \int P \ \partial x, \]
\[ Q'' = \int \frac{\partial^2 Q}{\partial x^2} \ dx = -\int u \ \partial^2 x, \]
\[ Q''' = \int \frac{\partial^3 Q}{\partial x^3} \ dx = \int u \ \partial^3 x, \]
\[ Q'''' = \int \frac{\partial^4 Q}{\partial x^4} \ dx = \int u \ \partial^4 x, \]
\[ Q''''' = \int \frac{\partial^5 Q}{\partial x^5} \ dx = \int u \ \partial^5 x, \]
\[ \text{etc.} \]

§. 36. Quod si jam nostrarum serierum, sive quod coddem redivit, quantitatum

\[ P, \ P', \ P'', \ P'''', \ \text{etc. et} \ Q, \ Q', \ Q'', \ Q''', \ \text{etc. eos}
\]

\[ \text{tantum valores desideramus, quos adipiscuntur posito } z = 1, \text{hoc}
\]

\[ \text{commodi assequimur, ut in formulis integralibus, ubi solus angulus}
\]

\[ u \text{ pro variabili habetur, statim ante integrationes ponere liceat } z = 1,
\]

\[ \text{hoc autem facto erit} \]

\[ P = \frac{\cos^3 u - 1}{2 - 2 \cos u} = -\frac{1}{1} \text{ et } Q = \frac{\sin^3 u}{2 - 2 \cos u} = \frac{1}{2} \cot^2 \frac{1}{2} u, \]

\[ \text{tum vero porro} \]

\[ \int P \ \partial u = A - \frac{1}{u}, \]
\[ \int \partial u / P \ \partial u = B - \frac{1}{u}, \]
\[ \frac{1}{2} \int \partial u / P \ \partial u = C + B u - \frac{1}{2} u, \]
\[ \frac{1}{4} \int \partial u / P \ \partial u = D + C u + \frac{1}{4} B u^2 - \frac{1}{8} u^3, \]

\[ \text{at pro formulis, ubi est } Q, \text{ calculus non tam concinne succedit;}
\]

\[ \text{erit enim} \]

\[ \text{etc.} \]

\[ \text{etc.} \]
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\[ Q = \frac{1}{4} \cot \frac{1}{2} u, \]
\[ \int Q \, d u = \frac{1}{2} \sin \frac{1}{2} u, \]
\[ \int \partial u \, Q \, d u = \int \partial u l \sin \frac{1}{2} u, \]

quae formula cum omnem integrationem suspit, vix ulterior progressi licet; interim tamen erit

\[ \int \partial u / \partial u / Q \, d u = \int \partial u / \partial u l \sin \frac{1}{2} u, \]
\[ \int \partial u / \partial u / Q \, d u = \int \partial u / \partial u l \sin \frac{1}{2} u. \]

§ 37. Quod ad priones formas variabilem \( z \) involventes attinet, per notas reductiones elicietur:

\[ \int \frac{P^2}{z} \, dz = \int \frac{P}{z} \, dz + \int \frac{P}{z} \, dz = \int \frac{P}{z} \, dz, \]

ubi prius membrum \( \int \frac{P}{z} \, dz \) evanesce posito \( \int = 1 \), tum vero

\[ \int \frac{P}{z} \, dz = \int \frac{P}{z} \, dz + \int \frac{P}{z} \, dz = \int \frac{P}{z} \, dz \cdot (\frac{1}{2})^2, \]

quibus expressionibus ulterior exhibitis colligimus fore

\[ P' = \int \frac{P}{z} \, dz, \]
\[ P'' = -\int \frac{P}{z} \, dz \cdot \frac{1}{2}, \]
\[ P''' = -\int \frac{P}{z} \, dz \cdot \frac{(1z)^2}{1+2}, \]
\[ P'''' = -\int \frac{P}{z} \, dz \cdot \frac{(1z)^3}{1+2}, \]

§ 38. Ex his igitur sequentium formulæ integralium valorem assignare possumus, casu quo \( z = 1 \),

\[ P = -\frac{1}{2}, \]
\[ P' = \int \frac{P}{z} \, dz = -\sin \frac{1}{2} u, \]
\[ P'' = -\int \frac{P}{z} \, dz = -B - Au - \frac{1}{4} u, \]
\[ P'''' = -\int \frac{P}{z} \, dz \cdot \frac{(1z)^2}{1+2} = \int \partial u / \partial u l \sin \frac{1}{2} u, \]

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\[ P^{''''} = \int p \frac{dz}{z} \cdot \frac{(iz)^5}{4 \cdot 2 \cdot 3} = D + Cu + \frac{1}{6} Bu + \frac{1}{6} Au^2 - \frac{5}{8} u^4, \]

\[ P^{V} = \int p \frac{dz}{z} \cdot \frac{(iz)^3}{4 \cdot 2 \cdot 3 \cdot 4} = \int \partial u \int \partial u \int \partial u \int \partial u \sin \frac{1}{2} u, \]

etc.

\[ Q = \frac{1}{2} \cot \frac{1}{2} u, \]

\[ Q' = \int \frac{Q \partial z}{z} = A - \frac{1}{2} u, \]

\[ Q'' = -\int \frac{Q \partial z}{z} \cdot \frac{1}{4} = \int \partial ul \sin \frac{1}{2} u, \]

\[ Q'''' = +\int \frac{Q \partial z}{z} \cdot \frac{(iz)^3}{6} = C - Bu - \frac{1}{6} Au + \frac{1}{12} u^6, \]

\[ Q'''' = -\int \frac{Q \partial z}{z} \cdot \frac{(iz)^6}{6} = \int \partial u / \partial u / \partial u / \partial u / \partial u \sin \frac{1}{2} u, \]

\[ Q^V = +\int \frac{Q \partial z}{z} \cdot \frac{(iz)^4}{2} = E - Du - \frac{1}{6} C u + \frac{1}{6} Bu + \frac{1}{2} Au^4 - \frac{5}{6} u^5, \]

etc.

§. 39. Cum igitur sit

\[ P = \frac{z \cos u - 2 \frac{dz}{z}}{1 - 2 z \cos u + z} \quad \text{et} \quad Q = \frac{z \sin u}{1 - 2 z \cos u + z}, \]

hactenus id sumus assecuti, ut harum duarum formularum integralium

\[ \int \frac{\partial z}{1 - 2 z \cos u + z} (iz)^n \quad \text{et} \quad \int \frac{\partial z}{1 - 2 z \cos u + z} (iz)^n \]

valores casu \( z = 1 \) commode per angulum \( u \) assignare valeamus, si modo constaret, quo facto quantitates \( A, B, C, D \), etc. determinari oporteat, id quod vix alio modo nisi per ipsas series, unde haec quantitates sunt natae, fieri posse videtur.

§. 40. Omisso igitur formulis integralibus, quae quantitatem \( Q \) involvant, quippe quarum integratio minus succedit, altreras tantum consideremus, et posito statim \( z = 1 \) ubi sit \( P = -\frac{1}{2} \), ita ut sit

\[ \cos u \rightarrow \cos 2u \rightarrow \cos 3u \rightarrow \cos 4u \rightarrow \text{etc.} = -\frac{1}{2}, \]
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si per \( \partial u \) multiplicemus et integremus, habebimus

\[ Q' = \frac{\sin u}{1} + \frac{\sin 2u}{2} + \frac{\sin 3u}{3} + \frac{\sin 4u}{4} + \frac{\sin 5u}{5} + \text{etc.} \quad = \ A - \frac{1}{2} u, \]

quae constans nihil aequalis videri potest, quia posito \( u = 0 \) summa seriei evanescere videtur; at sumto angulo \( u \) infinite parvo series praebebit

\[ u - u - u + u + u \quad \text{etc. et infinitum;} \]

notum autem est, talem seriem summam finitam habere posse, unde hoc casu omissae statuamus \( u = \pi \), seu potius \( u = \pi + \omega \), probatque haec series existente \( \omega \) angulo infinite parvo,

\[ - \omega + \omega - \omega + \omega - \omega + \text{etc.} \]

ubi, quia signa alternantur, nullum est dubium, quin summa seriei evanescat, quae cum esse debeat \( A - \frac{\pi}{2} \), evidens est, fieri constantem \( A = \frac{\pi}{2} \), ita, ut jam habeamus

\[ Q' = \frac{\sin u}{1} + \frac{\sin 2u}{2} + \frac{\sin 3u}{3} + \frac{\sin 4u}{4} + \frac{\sin 5u}{5} + \text{etc.} \quad = \frac{\pi - u}{2}. \]

Hoc modo constantem determinandi Illust. Daniel Bernoulli primus est usus, qui praeterea multa praeclera circa indolem harum serierum annotavit.

§. 41. Multiplicemus porro hanc ultimam seriem per \( -\partial u \),

et integrationi dabim

\[ P'' = \frac{\cos u}{1^2} + \frac{\cos 2u}{2^2} + \frac{\cos 3u}{3^2} + \frac{\cos 4u}{4^2} + \text{etc.} = B - \frac{\pi u}{2} + \frac{\pi u}{4}, \]

ad quam constantem inveniendam ponamus primo \( u = 0 \), fietque

\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = B. \]

Cujus seriei summam jam pridem primus demonstravi esse \( = \frac{\pi \pi}{6} \); verum si haec veritas nobis esset ignota, egregia illa methodo a magno Bernoulli adhibita utamur, ac ponamus \( u = \pi \) eritque

\[ - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \text{etc.} = B - \frac{\pi \pi}{2} + \frac{\pi \pi}{4} = B - \frac{\pi \pi}{4}; \]
ambae haec series additae dabunt
\[ \frac{3}{2^2} + \frac{3}{4^2} + \frac{2}{6^2} + \frac{2}{8^2} + \text{ etc.} = 2B - \frac{\pi \nu}{4}, \]
cujus duplum praebet
\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} = 4B - \frac{\pi \nu}{2} = B; \]
unde colligitur \( B = \frac{\pi \nu}{6}, \) ita ut sit
\[ P'' = \cos \frac{\nu}{4^2} + \cos \frac{2\nu}{2^2} + \cos \frac{3\nu}{3^2} + \cos \frac{4\nu}{4^2} + \text{ etc.} = \frac{\pi \nu}{6} - \frac{\nu \nu}{4} + \frac{\nu \nu}{4}. \]

§ 42. Eodem modo ulterior progrediamur, et denuo per \( \partial u \) multiplicando et integrando adipiscimur
\[ Q'' = \frac{\sin \frac{\nu}{4^2}}{4^2} + \frac{\sin \frac{2\nu}{2^2}}{2^2} + \frac{\sin \frac{3\nu}{3^2}}{3^2} + \frac{\sin \frac{4\nu}{4^2}}{4^2} + \text{ etc.} \]
\[ = C - \frac{\pi \pi u}{6} + \frac{\pi \nu}{4} + \frac{\nu \nu}{4}. \]
ubi si statuat \( u = 0 \), summa serici manifesto evanescit, prodiret
enim posit \( u = \omega \)
\[ \frac{\omega}{1^2} + \frac{\omega}{2^2} + \frac{\omega}{3^2} + \frac{\omega}{4^2} + \text{ etc.} = \frac{\omega \pi \nu}{6}, \]
qua \( \omega = 0 \) fit \( = 0 \), sicque est \( C = 0 \), ideoque
\[ Q'' = \frac{\sin \frac{\nu}{4^2}}{4^2} + \frac{\sin \frac{2\nu}{2^2}}{2^2} + \frac{\sin \frac{3\nu}{3^2}}{3^2} + \frac{\sin \frac{4\nu}{4^2}}{4^2} + \text{ etc.} = \frac{\pi \pi u}{6} - \frac{\pi \nu}{4} + \frac{\nu \nu}{4}. \]

§ 43. Ducatur haec series in \( - \partial u \), et integratio praebabit
\[ PV = \cos \frac{\nu}{4^2} + \cos \frac{2\nu}{2^4} + \cos \frac{3\nu}{3^4} + \cos \frac{4\nu}{4^4} + \text{ etc.} \]
\[ = D - \frac{\pi \nu u}{12} + \frac{\pi \nu}{4} + \frac{\nu \nu}{4}. \]
hinc sumto \( u = 0 \) fit
\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{ etc.} = D, \]
nunc vero fiat etiam \( u = \pi \), sitque
\[ - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \frac{1}{5^2} + \text{ etc.} = D - \frac{\pi \nu}{4}, \]
hae autem ambae series additae dant
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\[ \frac{2}{24} + \frac{2}{44} + \frac{2}{64} + \frac{2}{64} + \text{etc.} = 2D - \frac{\pi^4}{48}, \]

quae octies sumta ut numeratores fiant \( = 2\frac{1}{3} \), praebebit
\[ \frac{1}{24} + \frac{1}{34} + \frac{1}{34} + \frac{1}{44} + \text{etc.} = 16D - \frac{\pi^4}{6}, \]

unde oritur \( D = \frac{\pi^4}{90} \), quae est eadem summa seriei
\[ \frac{1}{24} + \frac{1}{34} + \frac{1}{44} + \text{etc.} \]

quam jam dudum inveneram, habeimus jam
\[ P = \frac{\cos u}{1^4} + \frac{\cos 2u}{2^4} + \frac{\cos 3u}{3^4} + \frac{\cos 4u}{4^4} + \text{etc.} \]
\[ = \frac{\pi^4}{90} - \frac{\pi^2 u^3}{12} + \frac{\pi u^5}{12} - \frac{u^7}{48} \]

§. 44. Multiplicando iterum per \( \partial u \) et integrando consequimur
\[ Q^V = \frac{\sin u}{1^4} + \frac{\sin 2u}{2^4} + \frac{\sin 3u}{3^4} + \frac{\sin 4u}{4^4} + \text{etc.} \]
\[ = E - \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{36} + \frac{\pi u^5}{48} - \frac{u^7}{240} \]

ubi uti in casu penultimo constans \( E \) iterum fit \( = 0 \), ita ut habeamus
\[ Q^V = \frac{\sin u}{1^4} + \frac{\sin 2u}{2^4} + \frac{\sin 3u}{3^4} + \frac{\sin 4u}{4^4} + \text{etc.} \]
\[ = \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{36} + \frac{\pi u^5}{48} - \frac{u^7}{240} \]

§. 45. Multiplicemus denuo per \( - \partial u \), probiditque integrando
\[ Q^V = \frac{\cos u}{1^4} + \frac{\cos 2u}{2^4} + \frac{\cos 3u}{3^4} + \frac{\cos 4u}{4^4} + \text{etc.} \]
\[ = F + \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{24} - \frac{\pi u^5}{120} + \frac{u^7}{720} \]

ubi ad constantem determinandum ponatur \( u = 0 \), eritque
\[ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = F, \]

tum vero sumatur \( u = \pi \), et fiet
\[ - \frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = F - \frac{\pi^4}{480} \].
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quaes additae dant
\[ \frac{2}{2^5} + \frac{2}{4^5} + \frac{2}{6^5} + \frac{2}{8^5} + \text{etc.} = \frac{2F}{480}, \]
quaem multiplicetur per 32, ut omnes numeratores sint \(64 = 2^6\);
et orietur
\[ \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = \frac{64F}{15} - \frac{\pi^6}{15} = F; \]
undecolligitur \( F = \frac{\pi^6}{15} \), ita ut sit
\[ pVI = \frac{\cos u}{4} + \frac{\cos 2u}{8} + \frac{\cos 3u}{3^4} + \frac{\cos 4u}{4^5} + \text{etc.,} \]
\[ = \frac{\pi^6}{945} + \frac{\pi^6}{90} \frac{u^2}{2} + \frac{\pi^6}{6} \frac{u^4}{24} + \frac{\pi^6}{120} + \frac{1}{2} \frac{u^6}{720}. \]

\[ \S\ 46. \] Has series ulteriorius continuare superflum foret, cum lex progressionis jam satis sit manifesta, praecluus si in subsidium vocentur summationes potestatum reciprocum parium, quas olim usque ad potestatem trigesimam supputatas dedi. Quod quo clarius perspicuatur, istae summas sequenti modo repraesentemus
\[ \frac{1}{4^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} = \alpha \pi \pi, \] ut sit \( \alpha = \frac{1}{8} \)
\[ \frac{1}{4^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} = \beta \pi \pi, \] ut sit \( \beta = \frac{1}{5} \)
\[ \frac{1}{4^5} + \frac{1}{5^5} + \frac{1}{6^5} + \text{etc.} = \gamma \pi \pi, \] ut sit \( \gamma = \frac{1}{54} \)
\[ \frac{1}{4^7} + \frac{1}{5^7} + \frac{1}{6^7} + \text{etc.} = \delta \pi \pi, \] ut sit \( \delta = \frac{1}{5463} \)
etc.

atque hic positis, sequentes habebimus integrationes, pro casu sci-
licet \( z = 1 \).

\[ Q' = \int \frac{\partial \xi \sin u}{1 - 2z \cos u + z^2} = \frac{\pi}{2} - \frac{\pi}{2} u = \text{Arc} \tan \frac{\sin u}{1 - \cos u}. \]

\[ P'' = -\int \frac{\partial z (\cos u - 2)}{1 - 2z \cos u + z^2} \frac{2}{4} = \alpha \pi \pi - \frac{1}{2} \pi u + \frac{1}{2} \frac{uu}{2}. \]

\[ Q''' = \int \frac{\partial z \sin u}{1 - 2z \cos u + z^2} \frac{2}{6} = \alpha \pi \pi \frac{u}{2} - \frac{1}{2} \pi u + \frac{1}{6} \frac{u^3}{2}. \]

\[ PIV = -\int \frac{\partial z (\cos u - 2)}{1 - 2z \cos u + z^2} \frac{1}{6} = \beta \pi^4 - \alpha \pi \pi \frac{u}{2} + \frac{1}{2} \pi \frac{u^3}{6} - \frac{1}{2} \frac{u^4}{24}. \]
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§ 47. Operae pretium erit, aliquos casus, quibus angulo $u$ datus valor tribuitur, ob oculos exponer. Ponamus igitur $u = 0$, quo casu formulae nostrae alternatim evanescunt, reliqua vero praebent:

\[ Q^{V} = -\int \frac{\partial x}{1-2z \cos u + \frac{z}{24}} \left( \frac{(iz)^4}{24} \right) = \frac{1}{6} \left( \frac{\pi}{\beta} \right)^4 - \frac{u}{6} \alpha \pi \pi \cdot \frac{u^6}{6} + \frac{1}{2} \pi \cdot \frac{u^6}{24} - \frac{1}{2} \pi \cdot \frac{u^6}{120} \]

\[ P^{VI} = -\int \frac{\partial x}{1-2z \cos u + \frac{z}{20}} \left( \frac{(iz)^4}{20} \right) = \gamma \pi^4 \cdot \frac{\pi}{\beta} \pi^4 \cdot \frac{u^4}{2} \pi \pi \cdot \frac{u^6}{24} \cdot \frac{1}{2} \pi \cdot \frac{u^6}{120} + \frac{1}{2} \pi \cdot \frac{u^6}{780} \]

\[ Q^{VII} = +\int \frac{\partial x}{1-2z \cos u + \frac{z}{20}} \left( \frac{(iz)^4}{80} \right) = \gamma \pi^4 \cdot \frac{\pi}{\beta} \pi^4 \cdot \frac{u^4}{6} + \alpha \pi \pi \cdot \frac{u^6}{480} + \frac{1}{2} \pi \cdot \frac{u^6}{780} + \frac{1}{2} \pi \cdot \frac{u^6}{5040} \]

etc. etc. etc.

§ 48. Hic notatu dignum occurrit, quod valores alterni, quos hic omisimus, etiam evanescant posito $u = \pi$; deinde non minus notatu dignum est, easdem formulas quaque evanescere posito $u = 2\pi$, sola prima excepta, quippe quae etiam non evanescit posito $u = 0$; reliquae vero, scilicet tercia, quinta, septima etc. certe evanescunt casibus $u = 0$ et $u = \pi$, quin etiam $u = 2\pi$. Quod quo clarius apparet, has formulas per factores repraesentemus, etrique tertiae valor

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\[
\frac{4}{15} u (\pi - u) (2\pi - u),
\]

quintae vero valor reperitur

\[
\frac{1}{3} (\pi - u) (2\pi - u) (4\pi + 6\pi u - 3u),
\]

quod etiam in sequentibus usu venit. In genere autem observari meretur, omnes nostras formulas sola prima excepta eosdem sortiri valores, sive ponatur \( u = 0 \) sine \( u = 2\pi \), quippe quibus tam idem sinus quam cosinus respondet. Videtur quidem eundem consensus locum habere debere, si ponatur \( u = 4\pi \) et \( u = 6\pi \), verum Illustr. Bernoullius jam luculenter ostendit, angulum \( u \) in his valoribus non ultra quatuor rectos augeri posse. Hujusmodi autem anomalia etiam in omnibus vulgaribus seriebus quibus arcus exprimitur occurrit; atque adeo in Leibniziana, in qua est

\[
u = \frac{\tan u}{1} - \frac{(\tan u)^2}{3} + \frac{(\tan u)^3}{5} - \frac{(\tan u)^4}{7} + \frac{(\tan u)^5}{9} - \text{etc.}
\]

angulum \( u \) non ultra 180 gr. augere licet. Si enim poneremus \( u = 180^\circ + u \), foret utique \( \tan u = \tan u \), neque tamen series illa exprimeret arcum \( \pi + u \) sed tantum arcum \( u \), cujusmodi phaenomena etiam in aliis similibus seriebus locum habent. Quod autem prima series hinc plerunque excipi debeat, ratio in eo est sita; quod in formula integrali posito \( u = 0 \) denominator fiat \((1 - \zeta)\), qui casu \( \zeta = 1 \) evanescit, ideoque formula in infinitum excrescit, id quod in sequentibus, quae per \( \zeta \) sunt multiplicatae, non amplius evenit, quia \( \frac{\zeta}{1 - \zeta} \) casu \( \zeta = 1 \) non amplius fit infinitus sed tantum \( = -1 \), et si major potestas logarithmi adsit, fit adeo \( = 0 \).

§. 49. Ponamus nunc etiam \( u = 90^\circ \), seu \( u = \frac{\pi}{2} \), ut sit \( \cos u = 0 \) et \( \sin u = 1 \), hocque casu omnes formulae generalis sequentes obtinebunt valores

\[
\int_{1 + \zeta^2}^{\frac{\pi}{2}} = \frac{\pi}{2}
\]
§. 50. Consideremus etiam casum \( u = 60^\circ \), sive \( u = \frac{\pi}{3} \), ut sit \( \cos u = \frac{1}{2} \) et \( \sin u = \frac{\sqrt{3}}{2} \), et formulae generales perducent ad sequentia integralia

\[
\begin{align*}
\frac{\sqrt{3}}{2} \int \frac{dx}{1 - x + x^2} &= \frac{\pi}{3} \\
\frac{1}{3} \int \frac{dx}{1 + x + x^2} \cdot (1 - 2x) &= \frac{\pi}{3} \\
\sqrt{3} \int \frac{dx}{1 + x + x^2} \cdot (1 - 2x) &= \frac{7\pi}{12} \\
\frac{2}{3} \int \frac{dx}{1 + x + x^2} \cdot (1 - 2x) &= \frac{7\pi}{10}
\end{align*}
\]

Simili modo si ponamus \( u = 120^\circ = \frac{2\pi}{3} \), ut sit \( \cos u = -\frac{1}{2} \) et \( \sin u = \frac{\sqrt{3}}{2} \), sequentes integrationes istis affines prodibunt

\[
\begin{align*}
\frac{\sqrt{3}}{2} \int \frac{dx}{1 + x + x^2} &= \frac{\pi}{3} \\
\frac{1}{3} \int \frac{dx}{1 + x + x^2} \cdot (1 - 2x) &= \frac{\pi}{15} \\
\sqrt{3} \int \frac{dx}{1 + x + x^2} \cdot (1 - 2x) &= \frac{2\pi^3}{81}
\end{align*}
\]

sicque pro lubitu numerus hujusmodi integrationum specialium augeri poterit.

§. 51. Quemadmodum istae integrationes memorabiles ex priore serie nostra \( P \) postis \( z = 1 \) sunt deductae, ita eodem modo alteram seriem \( Q \) pertractemus. Cum igitur sit

\( Q = \sin u \rightarrow \sin 2u \rightarrow \sin 3u \rightarrow \sin 4u \rightarrow \) etc. \( = \frac{1}{2} \cot \frac{1}{2} u \),

si per \( - \partial u \) multiplicemus et integremus, reperitur series

\( P' = \frac{\cos u}{1} + \frac{\cos 2u}{2} + \frac{\cos 3u}{3} + \frac{\cos 4u}{4} + \) etc. \( = \frac{1}{2} - l \sin \frac{1}{2} u + \Lambda \),

pro qua constante determinanda ponatur \( u = \pi \), ut sit

\[-1 + \frac{1}{3} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \) etc. \( = \Lambda ,
\]

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quocirca fit $A = -\Delta$, ita ut habeamus

$$P' = \frac{\cos u}{1} + \frac{\cos 2u}{2} + \frac{\cos 3u}{3} + \frac{\cos 4u}{4} + \text{etc.} = -\Delta \sin \frac{1}{4} u,$$

pro quo valore scribamus brevitas gratia $\Delta : u$, si quidem eum spectamus tanquam certam ipsius $u$ functionem, ita ut sit $P' = \Delta : u$.

§. 52. Multiplicando porro per $\partial u$ et integrando, nan-
ciscimus hanc, seriem

$$Q'' = \frac{\sin u}{4^2} + \frac{\sin 2u}{2^2} + \frac{\sin 3u}{3^2} + \frac{\sin 4u}{4^2} + \text{etc.} = \int \partial u \Delta : u = \Delta' : u;$$

ubi haec formula integralis involvet certam constantem, quam fa-
cile definire licet ex casu $u = 0$, quia ulla series evanesceit, fieri
debet $\Delta' : 0 = 0$, sicque integratio plene determinatur.

§. 53. Si eodem modo ulterius progrediamur, multiplicando
per $-\partial u$, probabit haec series

$$P'' = \frac{\cos u}{4^2} + \frac{\cos 2u}{2^2} + \frac{\cos 3u}{3^2} + \frac{\cos 4u}{4^2} + \text{etc.} = \int \partial u \Delta' : u = \Delta'' : u.$$  

Jam ad constantem, quae in hac expressione continetur, definien-
dam, sit $1^0 u = 0$, eritque

$$\frac{1}{4^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.} = \Delta'' : 0.$$

Sit $2^0. u = \pi$, et fit

$$-\frac{1}{4^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = \Delta'' : \pi,$$

quibus additis prodit

$$\frac{2}{5^2} + \frac{2}{4^2} + \frac{2}{3^2} + \frac{2}{8^2} + \text{etc.} = \Delta'' : \pi + \Delta'' : \pi,$$

hicque quatuor sumata erit

$$\frac{1}{4^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.} = 4\Delta'' : \phi + 4\Delta'' : \pi = \Delta'' : 0,$$

unde oritur

$$3 \Delta'' : 0 + 4 \Delta'' : \pi = 0;$$
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ex qua constans in formulam nostram integralem

\[ \Delta'' : u = - \int \partial u \Delta' \]

ingressa determinari debet.

§ 54. Multiplicemus denuo per \( \partial u \), et integrémus, pro-
dibitque

\[ Q^{IV} = \frac{\sin u}{4} + \frac{\sin 2u}{2} + \frac{\sin 3u}{3} + \frac{\sin 4u}{4} + \text{etc.} = \int \partial u \Delta'' : u = \Delta''' : u, \]

atque haec functio \( \Delta''' : u \) ita debet determinari, ut evanescat sum-
to \( u = 0 \), sive ut fiat \( \Delta''' : 0 = 0 \). Eodem modo ulterior pro-
grediendo fiat

\[ P^{IV} = \frac{\cos u}{4} + \frac{\cos 2u}{2} + \frac{\cos 3u}{3} + \frac{\cos 4u}{4} + \text{etc.} = - \int \partial u \Delta''' : u = \Delta''' : u, \]

hujusque functionis indeles sequenti modo determinabitur: ponatur
solicet ut hactenus \( u = 0 \), et \( u = \pi \), eritque

\[ \frac{1}{4} + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \text{etc.} = \Delta^{IV} : 0, \text{ et} \]

\[ -\frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \text{etc.} = \Delta^{IV} : \pi; \]

hinc addendo

\[ \frac{2}{4} + \frac{2}{4} + \frac{2}{4} + \frac{2}{4} + \text{etc.} = \Delta^{IV} : 0 + \Delta^{IV} : \pi, \]

et multiplicando per \( 16 \)

\[ \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \text{etc.} = 16 \Delta^{IV} : 0 + 16 \Delta^{IV} : \pi = \Delta^{IV} : 0, \]

sicque fieri debet

\[ 16 \Delta^{IV} : 0 + 16 \Delta^{IV} \pi = 0 \text{ etc.} \]

§ 55. Hinc igitur sequentes adipsemur integrationes pro casu \( z = 1 \)

I. \[ - \int \frac{\partial x (\cos u - z)}{1 - 2z \cos u + z^2} = - 2 \sin \frac{1}{2} u = \Delta : u \]

II. \[ \int \frac{\partial x \sin u}{1 - 2z \cos u + z^2} \frac{l z}{z} = \int \partial u \Delta u = \Delta' : u \]

III. \[ - \int \frac{\partial x (\cos u - z)}{1 - 2z \cos u + z^2} \frac{(l z)^2}{2} = - \int \partial u \Delta' u = \Delta'' : u \]
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IV. \( \int \frac{\partial x \sin u}{z - \partial x \cos u} \cdot \frac{(iz)^3}{6} = \int \partial u \Delta'' u = \Delta''' u \)

V. \( -\int \frac{\partial x (\cos u - z)}{z - \partial x \cos u} \cdot \frac{(iz)^4}{24} = -\int \partial u \Delta'' u = \Delta^{IV} u \)

VI. \( \int \frac{\partial x \sin u}{z - \partial x \cos u + x z} \cdot \frac{(iz)^6}{120} = \int \partial u \Delta^{IV} u = \Delta^{V} u \)

etc., etc., etc.

Has autem expressiones facile quousque libuerit continuare licet, si modo integratio cujusque integralis rite instituatur: conditiones autem, quas implieri oportet, sequenti modo referri possunt

\[
\begin{align*}
\Delta' : 0 & = 0 \\
\Delta'' : 0 & = 0 \\
\Delta^{IV} : 0 & = 0 \\
\Delta^{VI} : 0 & = 0 \\
\Delta^{VII} : 0 & = 0 \\
\Delta^{VIII} : 0 & = 0 \\
\end{align*}
\]

caeterum quia posteriores integrationes absolvere non licet, hinc parum utilitatis exspectare possimus.

§ 56. Caeterum methodus, qua hic sumus usi, ad constantes per quamque integrationem ingressas determinandas, a celeberrimo Bernoullio primum est exhibita, atque eo majori attensione digna est aestimanda, quod ejus ope summationes meae scripturarum reciprocum potestatum obtineri possunt, quandoquidem credideram, eas non aliter nisi ex consideratione infinitorum arcuum, qui vel eodem sinu vel cosinus gaudent, demonstrari posse.
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2). Comparatio valorum formulae integralis

\[ \int \frac{x^{p-t} \, dx}{\sqrt{(1-x^n)^{n-t}}} \]

a termino \( x = 0 \) usque ad \( x = 1 \) extensa. Nova Acta Acad. Imp. Scient. Petropolitanae. Tom. V. Pag. 86 − 117.

§ 57. In hac formula litterae \( n, p \) et \( q \) perpetuo designant numeros integros positivos, et pro quolibet numero \( n \) binis litteris \( p \) et \( q \) omnes valores tribui concipiuntur, ita ut hinc pro quovis numero \( n \) innumerae nascantur hujusmodi formulae integrales, quorum valores plurimas egregias relationes inter se servant; unde si eorum aliquot fuerint cogniti, reliquae omnes ex ipsis definhiri queant. Jam duobum equidem plures hujusmodi relationes demonstravi; cum autem hoc argumentum tum temporis nequit quam exhausisse, nunc accuratius in istas relationes inquirere constitui, et ejusmodi methodum adhibeo, quae omnes plane hujus generis relationes sit exhibitura; his enim inventis in numerabilia theorematata condi poterunt, quibus universa analysis non mediocris locupletari erit censenda.

§ 58. Quoniam igitur hoc modo pro quolibet numerò \( n \) ambae litterae \( p \) et \( q \) infinitos valores recipere possunt, ante omnia hic observari convenit, omnes hos innumerables casus semper ad numerum finimum revocari posse. Quantumvis enim magni numeri pro litteris \( p \) et \( q \) accipientur, eos casus semper, ad alios reducere licet, in quibus numeri \( p \) et \( q \) quantitate \( n \) futuri sint diminuti. Hoc igitur modo omnes hujusmodi casus tandem eo redigi poterunt, ut ambo numeri \( p \) et \( q \) infra exponentem \( n \) deprimantur; unde pro quolibet numero \( n \) eos tantum casus con-
siderasse sufficiet, quibus litterae $p$ et $q$ minores valores recipiant quam $n$, vel saltam hunc limitem non superent. Hoc igitur modo pro quovis numero $n$ multitudo casum, qui in computum veniunt, et quos inter se comparari oportet, prorsus eit determinata.

§. 59. Quemadmodum autem ista reductio litterarum $p$ et $q$ ad numeros continuo minores institui debeat, quamquam id satis in vulgus est notum, tamen ad formulam praesentem accommodasse juvabit. Statuatur silicet haec formula algebraica

$$x^p (1 - x^n)^{\frac{q}{n}} = V;$$

et quae

$$IV = p\, \partial x + \frac{q}{n} \left(1 - x^n\right),$$

hinc differentiando.

$$\frac{\partial V}{V} = \frac{p \, \partial x}{x} - \frac{q \, x^{n-1} \, \partial x}{1 - x^n} = \frac{p \, \partial x - (p + q) \, x^n \, \partial x}{x \, (1 - x^n)},$$

ubi si per $V$ multiplicemus, ac per partes integremus, orietur ista aequatio

$$V = p \int x^{p-1} \, \partial x \left(1 - x^n\right)^{\frac{q-n}{n}} - (p + q) \int x^{p+n-1} \, \partial x \left(1 - x^n\right)^{\frac{q-n}{n}}.$$

Quoniam igitur quantitas $V$ pro utroque integrationis termino evanescit, hinc adipiscimur istam reductionem

$$\int x^{p+n-1} \, \partial x \left(1 - x^n\right)^{\frac{q-n}{n}} = \frac{p}{p+q} \int x^{p-1} \, \partial x \left(1 - x^n\right)^{\frac{q-n}{n}},$$

o quis ergo reductionis ope exponens ipsius $x$ continuo quantitate $n$ diminuit poterit, donec tandem infra $n$ deprimatur.

§. 60. Deinde formula pro

$$\frac{\partial V}{V} = \frac{p \, \partial x - (p - q) \, x^n \, \partial x}{x \, (1 - x^n)}.$$
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\[
\frac{\partial V}{V} = \frac{(p - q) \, \partial x \left(1 - x^n\right) - q \, \partial x}{x (1 - x^n)},
\]

quae forma per \( V \) multiplicata ac denuo per partes integrata dabit

\[
V = (p + q) \int x^{p-1} \, \partial x \left(1 - x^n\right) - q \int x^{p-1} \, \partial x \left(1 - x^n\right)^{\frac{n}{n}}.
\]

unde quia posito \( x = 1 \) fit \( V = 0 \), oritur haec reductio

\[
\int x^{p-1} \, \partial x \left(1 - x^n\right)^{\frac{n}{n}} = \frac{q}{p - q} \int x^{p-1} \, \partial x \left(1 - x^n\right)^{\frac{n}{n}}.
\]

cujus reductionis ope exponens Binomii \( 1 - x^n \) unitate minuitur, sive quod eodem reedit, numerus \( q \) numero \( n \) imminuitur. Tali igitur reductione, quoties opus fuerit, repetita, exponens \( q \) tandem infra \( n \) deprimi poterit.

§ 61. Quoniam igitur pro quovis numero \( n \) ambos exponentes \( p \) et \( q \) tanquam minores quam \( n \) spectare licet, formulam hoc modo expressam repraesentemus

\[
\int \frac{x^{p-1} \, \partial x}{V(1 - x^n)^{n-q}}.
\]

Hic scilicet pro quovis numero \( n \) sufficit litteris \( p \) et \( q \) omnes valores ipso \( n \) minores tribuisses, quo pacto multitudo omnium casuum ad quemlibet exponentem \( n \) pertinentium ad numerum suis modicum reducetur, qui tamen eo major evadit, quo major fuerit exponentes \( n \).

§ 62. Multo magis autem numerus casuum diversorum diminuetur, si perpendamus, ambas litteras \( p \) et \( q \) inter se permutari posse, ita ut hujus formule

\[
\frac{x^{q-1} \, \partial x}{V(1 - x^n)^{n-p}}.
\]
SUPPLEMENTUM V.

valor ab illo prorsus, non disperepet. Ad quod ostendendum ponamus
\[ \int \frac{x^{p-1} \, dx}{\sqrt[n]{1 - x^n}^{n-q}} = S, \]
si scilicet ista formula integralis ab \( x = 0 \) usque ad \( x = 1 \) extendatur. Jam faciamus \( 1 - x^n = y^n \), ut formula sit
\[ S = \int \frac{x^{p-1} \, dx}{y^{n-q}}; \]
tum vero quia \( x^n = 1 - y^n \), erit \( x = (1 - y^n)^{\frac{1}{n}} \), hincque
\[ x^p = (1 - y^n)^{\frac{p}{n}}, \]
unde differentiando fit
\[ px^{p-1} \, dx = -py^{n-1} \, dy (1 - y^n)^{\frac{p-n}{n}}, \]
quo valore substituto erit
\[ S = - \int y^{q-1} \, dy (1 - y^n)^{\frac{p-n}{n}}, \]
quam formulam ab \( x = 0 \) usque ad \( x = 1 \), hoc est ab \( y = 1 \) usque ad \( y = 0 \), extendi oportet; permutatis igitur his terminis erit
\[ S = \int \frac{y^{q-1} \, dy}{\sqrt[n]{(1 - y^n)^{n-p}}} \left[ \text{ab } y = 0 \right] \left[ \text{ad } y = 1 \right]. \]
Sicque demonstratum est ambas litteras \( p \) et \( q \) semper inter se esse permutabiles.

§. 68. His praemissis, quo calculos sequentes magis in compendium redigere liceat, loco formulæ hujus integralis
\[ \int \frac{x^{p-1} \, dx}{\sqrt[n]{1 - x^n}^{n-q}} = \int \frac{x^{q-1} \, dx}{\sqrt[n]{(1 - x^n)^{n-p}}}. \]
scribamus hune characterem \((p, q)\), ubi perinde est, sive \(p\) ante \(q\), sive \(q\) ante \(p\) collocetur; semper autem hic certus exponens \(n\) subintelligi debet. Hic autem duo casus prae reliquis maxime memorabiles occurrunt. Prior casus est, quo numerorum \(p\) et \(q\) alterutri ipsi exponenti \(n\) est aequalis; si enim fuerit \(q = n\), erit ex priore formula \((p, n) = \int x^{p-1} \partial x = \frac{x}{p}\), sicque perpetuo habebimus \((p, n) = \frac{x}{p}\), hincque etiam \((n, q) = \frac{x}{q}\). Alter casus notatu dignissimus locum habet, quando \(p + q = n\), quo casu semper est

\[
(p, q) = \frac{n}{n \sin \frac{p\pi}{n}} = \frac{n}{n \sin \frac{q\pi}{n}}.
\]

Ad hoc ostendendum sit \(q = n - p\), hincque formula propuesta \(\int \frac{x^{p-1} \partial x}{\sqrt[1]{(1 - x^n)^p}}\), tum ponatur \(\frac{x}{\sqrt[1]{(1 - x^n)}^p} = z\), et quia \(\frac{x^p}{\sqrt[1]{(1 - x^n)}} = z^p\), erit \(S = \int \frac{x^p \partial x}{x}\). Ex facta autem positione sequitur \(x^n = \frac{z^n}{1 + z^n}\), hincque

\[
l n x = n l z - l (1 + z^n),
\]

ergo differentiando

\[
\frac{\partial x}{x} = \frac{\partial z}{z} - \frac{z^{n-1} \partial z}{1 + z^n} = \frac{\partial z}{z (1 + z^n)},
\]

ita ut jam sit

\[
S = \int \frac{z^{p-1} \partial z}{1 + z^n}.
\]

Quia autem sumto \(x = 0\) fit etiam \(z = 0\), at vero sumto \(x = 1\) prodit \(z = \infty\), hoc integrale a termino \(z = 0\) usque...
§ 64. Progrediamur nunc ad ipsum fundamentum, unde omnes relationes, quas quacumque, derivari convenit, et quod reductioni priori innititur; unde fit

\[ \int \frac{x^{p-1}}{\sqrt[2]{1-x^n}} \, \mathrm{d}x = \frac{p+q}{p} \int \frac{x^{n+p-1}}{\sqrt[2]{1-x^n}} \, \mathrm{d}x, \]

ubi loco \( \sqrt[2]{1-x^n} \) scribamus \( X \), ut sit

\[ \int \frac{x^{p-1}}{X} \, \mathrm{d}x = \frac{p+q}{p} \int \frac{x^{n+p-1}}{X} \, \mathrm{d}x \]

(hinc jam similis modo, si loco \( p \) scribamus \( n+p \), erit)

\[ \int \frac{x^{n+p-1}}{X} \, \mathrm{d}x = \frac{n+p+q}{n+p} \int \frac{x^{2n+p-1}}{X} \, \mathrm{d}x, \]

hincque sequitur fore

\[ \int \frac{x^{p-1}}{X} \, \mathrm{d}x = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \int \frac{x^{2n+p-1}}{X} \, \mathrm{d}x. \]

Quodsi similis modo ultimus progrediamur, perveniemus ad hanc equationem

\[ \int \frac{x^{p-1}}{X} \, \mathrm{d}x = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \int \frac{x^{2n+p-1}}{X} \, \mathrm{d}x. \]

Quare si hoc modo in infinitum progrediamur, habebimus

\[ \int \frac{x^{p-1}}{X} \, \mathrm{d}x = \frac{p+q}{p} \cdot \frac{n+p+q}{n+p} \cdot \frac{2n+p+q}{2n+p} \cdot \frac{1}{2n+p} \int \frac{x^{2n+p-1}}{X} \, \mathrm{d}x, \]

ubi \( i \) denotat numerum infinite magnum.
§. 66. Quodsi jam loco p alium quemcunque numerum r, pariter ipso n minorem, assumamus, erit simil modo
\[ \int \frac{x^{n-1}}{X} \, \partial x = \frac{r+q}{r} \cdot \frac{n+r+q}{n+r} \cdot \frac{2n+r+q}{2n+r} \times \]
\[ \times \frac{n+r+q}{n+r} \int x^{(i+1)n+r-1} \, \partial x \]
ubi littera i eundem numerum infinitum designat, ita ut utrinque idem factorum numerus adsit. Dividamus jam primum expressionem per istam, et quoniam extremae formule integralles, ex litteras p et r prae (i+1) n evanescentes, pro aequalibus inter se sunt habendi, facta divisione per singulos factores reperiemus hanc equationem
\[ \int \frac{x^{p-1}}{x^{r-1}} \, \partial x : X = \frac{r+q}{r} \cdot \frac{n+r}{n+p+q} \cdot \frac{n+p}{n+r+q} \times \]
\[ \times \frac{2n+r}{2n+p} \frac{2n+p+q}{2n+r+q} \frac{3n+r}{3n+p+q} \frac{3n+p}{3n+r+q} \times \text{etc.} \]
Restituantur jam loco harum formularum integralium characteres ante stabilitos, atque adippescemur istam relationem notatu dignissimam
\[ \frac{p}{r,q} = \frac{r+p+q}{2n+r+q} \cdot \frac{n+r}{2n+p+q} \cdot \frac{2n+p}{3n+r+q} \cdot \text{etc.} \]
quod productum ex infinitis membris componitur, quorum singula sunt fractiones, quaquem tam numeratores quam denominatores ex binis factoribus constant. Hos factores singulos eodem numero n augeri oportet, dum a quovis membro ad sequens progressimur, unde sufficit solum primum productum nosse, quod ergo ita reprehensabimus
\[ \frac{p,q}{r,q} = \frac{r+p+q}{r+q} \frac{2n+p+q}{2n+r+q} \frac{3n+p+q}{3n+r+q} \times \text{etc.} \]
SUPPLEMENTUM V.

§. 66. Quoniam litterae \( p \) et \( q \) nobis numeros quasi indennis significant, utamur litteris alphabeta initialibus ad numeros determinatos designandos, eritque eodem modo

\[
\begin{align*}
(a, b) &= \frac{a(a+b)}{a} \frac{(n+a)(n+a+b)}{(n+a)(n+a+b)} \\
(a, b) &= \frac{a(a+b)}{a} \frac{(n+a)(n+a+b)}{(n+a)(n+a+b)} \text{ etc.}
\end{align*}
\]

Hic jam loco \( \alpha \) scribamus \( \alpha \| c \), et productum infinitum hanc indue formam

\[
\begin{align*}
(a, b) &= \frac{(a+c)(a+b)}{a(a+c+b)} \frac{(n+a+c)(n+a+b)}{(n+a)(n+a+c+b)} \text{ etc.}
\end{align*}
\]

in quo producto ambae litterae \( b \) et \( c \) manifesto permutari possunt, unde idem productum infinitum etiam exprimet valorem hujus formae \( \frac{(a, c)}{(a+b, c)} \), unde sequitur ista aequalitas maxime memorablis \( \frac{(a, b)}{(a+c, b)} = \frac{(a, c)}{(a+b, c)} \); fractionibus igitur sublatis habebimus ilud insigne theorema

\[
(a, b) (a+b, c) = (a, c) (a+c, b);
\]

hucque theoremati universa analysis, qua utemur, erit superstructa.

§. 67. Cùm ob rationes supra allegatas numeri \( p \) et \( q \) exponentem \( n \) superare non debeant, etiam in forma theorematis modo allati singuli termini ibi occurrentes, qui sunt \( a, b, c, \alpha + b \) et \( \alpha + c \), quovis casu exponentem \( n \) superare non debent, sique nec \( \alpha + b \), necque \( \alpha + c \) major capi poterit quam \( n \). Hic autem primo observo litteras \( b \) et \( c \), inter se inaequalis statui debere: si enim esset \( c = b \), aequalitas in theoremate expressa foret identica; hanc ob rem perpetuo assumemus \( b > c \), ita ut maximus terminus in theoremate sit \( \alpha + b \), quem ergo exponentem \( n \) quovis casu excedere non oportet, quamobrem evolutionem formae generalis in theoremate contentae ita in classes distribuamus, quae inter se per maximum valorem termini \( \alpha + b \) distinguantur. Cum igitur nulla litterarum \( a, b, c \) nihil aequalis sumi quet, ac esse debet \( b > c \), minimus valor, quem
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terminus $a + b$ recipere potest, erit 3, in quo ergo primam classem constitueamus; sequentes vero classes constituenetur, dum termino $a + b$ valores 4, 5, 6, 7, etc. tribuantur.

I. Evolutio classis
qua $a + b = 3$.

§ 68. Hic ergo necessario erit $a = 1$, $b = 2$ et $c = 1$; ita ut hic nulla varietas locum inveniat, unde theorema nostrum supeditat hanc unicum relationem $(1, 2) (3, 1) = (1, 1) (2, 2)$. Dummodo igitur exponens $n$ non fuerit minor quam 3, semper haec insignis relatio locum habet

$$\int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx \quad \int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx = \int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx \cdot \int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx$$

quae forma, quia in qualibet charactere terminos inter se permutare licet, etiam hoc modo representandi poterit:

$$\int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx \cdot \int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx = \int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx \cdot \int_{\frac{1}{\sqrt[2]{1-x^n}}}^{1} \frac{x}{\sqrt[2]{1-x^n}} \, dx$$

II. Evolutio classis
qua $a + b = 4$.

§ 69. Quoniam $b$ binario minor esse nequit, hic erit vel $b = 2$, vel $b = 3$. Sit igitur primo $b = 2$, etque $a = 2$ et $c = 1$; unde ex nostro theoremate sequitur haec relatio $2, 2) (4, 1) = (2, 1) (3, 2)$, quae forma manifesto oritur ex classe prima, si ibi termini priores cujusque characteris unitate anguntur; id quod etiam inde intelligere licet, quod omnes termini priores litteram $a$ continent, qua unitate aucta processus semper fit ad classem sequentem.
§ 70. Deinde vero hic quoque statui potest \( b = 3 \), unde fit \( a = 1 \); at vero littera \( c \) jam duos valores, vel \( 1 \), vel \( 2 \) sortiri poterit; priore casu, quo \( c = 1 \), probit ista aequatio \( (1,3) (4, 1) = (1,1) (2, 3) \); alter vero casus, quo, \( c = 2 \), praebet hanc aequationem \( (1,3) (4, 2) = (1, 2) (3,3) \). Sicque haec classis omnino sequentes tres relationes continebit

\[
\begin{align*}
1^o. \ (2,2) (4,1) & = (2,1) (3,2), \\
2^o. \ (1,3) (4,1) & = (1,1) (2,3), \\
3^o. \ (1,3) (4,2) & = (1,2) (3,3).
\end{align*}
\]

III. Evolutio classis
qua \( a - b = 5 \).

§ 71. In hac igitur classe primo occurrent tres relationes praecedentes, si modo termini priores cujusque characteris unitate augeantur: hinc enim casus exsurgent, quibus est vel \( b = 2 \), vel \( b = 3 \). De novo igitur hic accedent casus, quibus \( b = 4 \) et \( a = 1 \), ubi ergo erit vel \( c = 1 \), vel \( c = 2 \), vel \( c = 3 \), quibus igitur tribus casibus evolutis omnino in hac classe sex continebuntur relationes, quae erunt

\[
\begin{align*}
1^o. \ (3,2) (5,1) & = (3,1) (4,2), \\
2^o. \ (2,3) (5,1) & = (2,1) (3,3), \\
3^o. \ (2,3) (5,2) & = (2,2) (4,3), \\
4^o. \ (1,4) (6,1) & = (1,1) (2,4), \\
5^o. \ (1,4) (5,3) & = (1,2) (3,4), \\
6^o. \ (1,4) (5,3) & = (1,3) (4,4).
\end{align*}
\]

IV. Evolutio classis
qua \( a + b = 5 \).

§ 72. Hic igitur primum occurrent omnes relationes proxime praecedentes, si modo termini priores cujusque cha-
xacteris unitate augeantur: hi scilicet nascuntur, si fuerit vel $b = 2$, vel $b = 3$, vel $b = 4$. Praeterea vero insuper accedent casus $b = 5$ et $a = 1$, ubi littera $c$ recipere poterit valores 1, 2, 3, 4, sicque, omnino in hac classe occurrent decem relationes sequentes

\[
\begin{align*}
1^o. (4, 2) (6, 1) & = (4, 1) (5, 2), \\
2^o. (3, 3) (6, 1) & = (3, 1) (4, 3), \\
3^o. (3, 3) (6, 2) & = (3, 2) (5, 2), \\
4^o. (2, 4) (6, 1) & = (2, 1) (3, 4), \\
5^o. (2, 4) (6, 2) & = (2, 2) (4, 4), \\
6^o. (2, 4) (6, 3) & = (2, 3) (5, 4), \\
7^o. (1, 5) (6, 1) & = (1, 1) (2, 5), \\
8^o. (1, 5) (6, 2) & = (1, 2) (3, 5), \\
9^o. (1, 5) (6, 3) & = (1, 3) (4, 5), \\
10^o. (1, 5) (6, 4) & = (1, 4) (5, 5),
\end{align*}
\]

V. Evolutio classis

qua $a = b = 7$.

§ 73. Hic igitur primo occurrent omnes relationes classis IV. postquam scilicet omnes terminos priores singulorum characterum unitate auxerimus, quos igitur hic apposuisse non erit necessae, ac sufficiat eas tantum relationes hic exponere, quae de novo accedunt et ex valore $b = 6$ oriuntur, existente $a = 1$; ubi pro $c$ sumi poterunt numeri 1, 2, 3, 4, 5, ita ut harum numeros sit quinque. Haec ergo relationes sunt

\[
\begin{align*}
(1, 6) & (7, 1) = (1, 1) (2, 6), \\
(1, 6) & (7, 2) = (1, 2) (3, 6), \\
(1, 6) & (7, 3) = (1, 3) (4, 6), \\
(1, 6) & (7, 4) = (1, 4) (5, 6), \\
(1, 6) & (7, 5) = (1, 5) (6, 6).
\end{align*}
\]

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VI. EVOLUTIO CLASSIS

qua \( a + b = 8 \).

§ 74. In hac jam classe primo occurrent omnes decem relationes classis IV., dum scilicet omnes termini priores binario augeantur; praeterea quoque accedent quinque relationes in classe V allatae, dum partes priores unitate augebuntur; praeter has vero de novo accedent 6 sequentes relationes ex valoribus \( a = 1 \) et \( b = 7 \) oriundae, dum litterae c valores 1, 2, 3, 4, 5, 6 ordine tribuuntur, quae ergo erunt

\[
\begin{align*}
(1, 7) (8, 1) &= (1, 1) (2, 7) \\
(1, 7) (8, 2) &= (1, 2) (3, 7) \\
(1, 7) (8, 3) &= (1, 3) (4, 7) \\
(1, 7) (8, 4) &= (1, 4) (5, 7) \\
(1, 7) (8, 5) &= (1, 5) (6, 7) \\
(1, 7) (8, 6) &= (1, 6) (7, 7).
\end{align*}
\]

VII. EVOLUTIO CLASSIS

qua \( a + b = 9 \).

§ 75. Ut omnes relationes ad hanc classem pertinentes adipiscamur, notandum est primo hic occurrere decem relationes classis IV., dum partes priores ternario augeantur. Secundo adiici oportet quinque relationes in classe V exhibitas, ubi partes priores binario augeri debent. Tertio huc referri debent sex relationes classis VI., partes priores unitate augendo. Insuper vero de novo accedent septem relationes ex valoribus \( a = 1 \) et \( b = 8 \) natae, dum litterae c tribuuntur ordine valores 1, 2, 3, 4, 5, 6, 7. Hae relationes sunt

\[
\begin{align*}
(1, 8) (9, 1) &= (1, 1) (2, 8) \\
(1, 8) (9, 2) &= (1, 2) (3, 8) \\
(1, 8) (9, 3) &= (1, 3) (4, 8)
\end{align*}
\]
§ 76. Hinc jam ordo progressionis tam clare perspicitur, ut superfluum foret has evolutiones ulterius prosequi; quando- quidem ob ingentem multitudinem relationum, quae in sequentibus classibus occurrerent, nimis molestam foret omnes percurrere. Quin etiam nostrum institutum vix permittere videtur, ut in nostra formula generali exponentem \( n \) ultra sex vel septem angeamus, si quidem omnes relationes ad cum pertinentes enumerare voluerimus. Sin autem animus sit aliquas tantum expendere, classes allatae ab- unde sufficiunt, dum termini priores cujusque classis quovis numero augebuntur.

§ 77. His jam classibus expeditis, formulam integralem propositam

\[
\int \frac{x^{p-1} \partial x}{\sqrt[1-n]{(1-x^n)^{n-1}}} \text{ secundum diversos valores exponentis } n \text{ pertactemus, dum scilicet successive assumamus } n = 3, \ n = 4, \ n = 5, \text{ etc. et pro quolibet ordine omnes relationes, quae in eo occurrere possunt, expendamus. Evidens autem est, quicun- que numerus exponenti } n \text{ tribuatur, formulæ omnium classum inferiorum, in quibus scilicet terminus } a \rightarrow b \text{ non superet } n, \text{ in usum vocari posse. Ex quo intelligitur, si fuerit } n = 3 \text{unicam relationem locum invenire; statim autem ac } n \text{ magis augetur, numerus omnium relationum mox ita increcit, ut nimis molestum foret om- nes recensere. Hosigitur diversos ordinem, ex exponente } n \text{ con- stituendos, a primo incipiendo, ordine involvamus.} \]
ORDO I.

quo \( n = 3 \) et formula

\[
(p, q) = \int \frac{x^{p-1} \partial x}{\sqrt[3]{1 - x^3}^{3-q}} = \int \frac{x^{q-1} \partial x}{\sqrt[3]{1 - x^3}^{3-p}}.
\]

§. 78. Cum hie sit \( n = 3 \), erit \((8, 4) = 1\); formulae autem integrales hujus ordinis erunt tres; scilicet \( 1^0 \), \((1, 4)\), \(2^0 \cdot (1, 2), 3^0 \cdot (2, 2)\), quarum media, ob \( 1 + 2 = 3 \), a circulo pendet, quae ergo, quia est cognita, ponatur

\[
(1, 2) = \frac{\pi}{3 \sin \frac{\pi}{3}} = \frac{2 \pi}{3 \sqrt{3}} = A.
\]

Hic igitur tantum classis prima locum habet, quae nobis hanc unicum aequationem suppediit \( A = (1, 1) \cdot (2, 2) \).

§. 79. Hinc ergo patet, productum ex binis formulis transcendentibus \((1, 1)\) et \((2, 2)\) aequali quantitati circulari \( A = \frac{2 \pi}{3 \sqrt{3}} \), ita ut pro ipsis formulis integralibus habeamus hanc relationem

\[
\int \frac{\partial x}{\sqrt[3]{1 - x^3}^2} \cdot \int \frac{x \partial x}{\sqrt[3]{1 - x^3}} = \frac{2 \pi}{3 \sqrt{3}};
\]

unde si altera harum duarum formularum fuerit cognita, etiam valor alterius assignari potest. Spectemus ergo priorem quasi nobis esset cognita, etiam si sit transcendentis, eamque ponamus

\[
(1, 1) = \int \frac{\partial x}{\sqrt[3]{1 - x^3}^2} = P,
\]

eritque \((2, 2) = \frac{A}{P}\). Sicque nihil praeterea in hoc ordine notandum relinquitur.
AD TOM. I., CAP. VIII.

Ord o II.

quo \( n = 4 \) et formula

\[
(p, q) = \int \frac{x^{n-1} \, \text{d}x}{\sqrt[4]{1-x^4}} = \frac{x^{n-1} \, \text{d}x}{2 \sqrt[4]{1-x^4}}
\]

§ 80. Cum igitur hic sit \( n = \frac{4}{2} \), exit \((4, 1) = 1\) et \((4, 2) = \frac{3}{2} \); formulae autem integrales ad hunc ordinem pertinentes erunt sex sequentes: 1\(^\circ\), \((1, 1)\); 2\(^\circ\), \((1, 2)\); 3\(^\circ\), \((1, 3)\), 4\(^\circ\), \((2, 2)\), 5\(^\circ\), \((2, 3)\), 6\(^\circ\), \((3, 3)\), inter quas ergo reperintur duae formulae circulares \((1, 3)\) et \((2, 2)\), quas propterea litteris \(A\) et \(B\) designemus, ponendo

\[
(1, 3) = \frac{\pi}{4 \sin \frac{\pi}{4}} = \frac{\pi}{2} = A, \text{ et}
\]

\[
(2, 2) = \frac{\pi}{4 \sin \frac{2\pi}{4}} = \frac{\pi}{4} = B,
\]

ita ut sit \(\frac{A}{B} = \sqrt{2}\).

§ 81. In hoc ergo ordine aequationes tam primae quam secundae classis locum habere possunt; secunda autem classis nobis has tres praebet aequationes

1\(^\circ\), \(B = (2, 1)(3, 2)\), 2\(^\circ\), \(A = (1, 1)(2, 3)\), 3\(^\circ\), \(A = 2(1, 2)(3, 3)\),

classis vero prima insuper dat hanc aequationem \(A(1, 2) = (1, 1)B\), sive \(\frac{A}{B} = \frac{(1, 1)}{(1, 2)}\) quae autem aequatio jam ex duabus prioribus deductur; namque ob \((3, 2) = (2, 3)\), secunda per primam divisa dabit \(\frac{A}{B} = \frac{(1, 1)}{(1, 2)} = \sqrt{2}\); ita ut ratio inter has duas formulas sit algebraica, quae ergo imprimit notari meretur

\[
\int \frac{\, \text{d}x}{\sqrt[4]{1-x^4}} : \int \frac{\, \text{d}x}{\sqrt[4]{1-x^4}} = \sqrt{2}.
\]
SUPPLEMENTUM V.

§. 82. Jam in hoc ordine, praeter binas formulas circulares, \((1, 3) = A\) et \((2, 2) = B\); tanquam cognitam etiam introducamus formulam \((1, 2)\), quae in ordine praeecedente erat circularis, nunc autem est transcendens, ex quibus ponamus \((1, 2) = \frac{\pi x}{2} = P\); ubi caveatur, ne litterae A et P cum iis confundantur, quibus in formulis praeecedentibus sumus usi, id quod etiam de ordinibus sequentibus est tenendum. His igitur litteris introductis aequationes nostrae erunt sequentes tres

\[ 1^o. \quad B = P(3, 2), \quad 2^o. \quad A = (1, 1)(2, 3), \quad 3^o. \quad A = 2P(3, 3), \]

quandoquidem vidimus, quartam in praeecedentibus jam contineri.

§. 83. Ope harum trium aequationum ergo ternas formulas integralis etiamnunc incognitas per ternas A, B et P, quas ut datas spectamus, determinare licebit. Ex prima enim fit \((3, 2) = \frac{A}{2P}\); ex tertia autem fit \((3, 3) = \frac{A}{2P}\); tum vero ex secunda colligitur \((1, 1) = \frac{A}{(3, 2)} = \frac{AP}{B}\). Cum igitur in hoc ordine omnino sint sex formulae integrales, earum ternae per tres reliquas definiri possunt, quas determinationes igitur ob oculos posuisse juvabit

\[
\begin{align*}
(1, 3) & = A = \frac{\pi}{2} \sqrt{2}; \\
(2, 2) & = B = \frac{\pi}{2}; \\
(1, 2) & = P = \frac{\pi x}{2} \\
(1, 1) & = \frac{AP}{B}; \\
(2, 3) & = \frac{B}{P}; \\
(3, 3) & = \frac{A}{2P}.
\end{align*}
\]

Ex postremis ergo erit

\[(2, 8) : (3, 3) = 2B : A = \sqrt{2} : 1,\]
ita ut etiam hae duae formulæ inter se habeant rationem algebraicam, qua est
\[ \int \frac{x \, x \, dx}{\sqrt{(1 - x^2)}} = 2 \int \frac{x \, dx}{\sqrt{(1 - x^2)}}. \]
Aliis insignibus relationibus, utpote satis cognitis, hic non immoramus.

Ord. III.

quod \( n = 5 \) et formula
\[
(p, q) = \int \frac{x^{p-1} \, dx}{\sqrt{(1 - x^5)^{n-q}}} = \int \frac{x^{q-1} \, dx}{\sqrt{(1 - x^5)^{n-p}}}
\]

§ 84. Hic igitur ob \( n = 5 \) ante omnia erit
\[(5, 1) = 1, \quad (5, 2) = \frac{1}{5}, \quad (5, 3) = \frac{3}{5}, \]
formulae autem integrales hujus ordinis erunt hae decem
1º. \((1, 1)\), 2º. \((1, 2)\), 3º. \((1, 3)\), 4º. \((1, 4)\), 5º. \((2, 2)\),
6º. \((2, 3)\), 7º. \((2, 4)\), 8º. \((3, 3)\), 9º. \((3, 4)\), 10º. \((4, 4)\),
inter quas quarta et sexta sunt circulares, quas ergo ita designamus
\[(4, 4) = \frac{\pi}{5 \, \sin \frac{\pi}{5}} = A \text{ et } (2, 3) = \frac{\pi}{5 \, \sin \frac{2 \, \pi}{5}} = B. \]

Praeterea vero binas formulæ, quae in ordine praecedenti erant circulares, nunc autem sunt transcendentes, etiam peculiariibus literis notemus, solumet \((1, 3) = P \text{ et } (2, 2) = Q. \) Mox enim patet,
dummodo etiam istae formulæ tanquam cognitae spectentur reliquas sex omnes per has quatuor determinari posse.
SUPPLEMENTUM V.

§. 85. Quoniam hic tres classes priores locum habere possunt, consideremus primo aequationes, quas tertia classis suppeditat, et quae introductis his valoribus erunt

1o. \( B = P(4, 2) \),

2o. \( B = (2, 4)(3, 3) \),

3o. \( B = 2Q(4, 3) \),

4o. \( A = (1, 1)(2, 4) \),

5o. \( A = 2(1, 2)(3, 4) \),

6o. \( A = 3P(4, 4) \).

Quas hoc modo succinctius repraesentare licet

\[ A = (1, 1)(2, 4) = 2(1, 2)(3, 4) = 3P(4, 4), \]
\[ B = P(4, 2) = (2, 1)(3, 3) = 2Q(4, 3); \]

ubi sex occurrunt producta ex binis formulis integralibus, quae singula quantitati circulari aequantur, unde totidem egregia theorema formari possent, nisi hinc jam clare in oculos incurrerent.

§. 86. Jam videamus, quot formulas integrales incognitas ex quatuor cognitis \( A, B, P \) et \( Q \) definire queamus, at vero prima dat \( (4, 2) = B_P \), tertia praebet \( (4, 3) = \frac{B}{2Q} \), sexta dat \( (4, 4) = \frac{A}{5P} \); hinc autem porro ex quarta deducimus

\[ (1, 1) = \frac{A}{(2, 4)} = \frac{A_P}{B}, \]

ex quinta vero deducimus

\[ (1, 2) = \frac{A}{(3, 4)} = \frac{A_Q}{B}. \]

Denique ex secunda elicimus.

\[ (3, 3) = \frac{B}{(2, 1)} = \frac{BB}{A_Q}, \]

sicque ex his sex aequationibus sex determinationes sumus adepti; atque adeo per litteras \( A, B, P \) et \( Q \) valores omnium reliquarum litterarum assignavimus.
§ 87. Quoniam igitur hactenus tantum classe tertia sumus usi, consideremus etiam aequationes secundae classis, quae sunt

1°. \( A \frac{Q}{B} = B \) (2, 1),

2°. \( A \frac{P}{B} = B \) (1, 1), et

3°. \( P \) (4, 2) = (1, 2) (3, 3);

verum si hic valores modo inventos substituimus, aequationes mere identicae resultant, ita ut hinc nulla nova determinatio sequatur. Idem usu venit ex aequatione prima classis, quae erat (2, 1) (3, 1) = (1, 1) (2, 2), quae facta substitutione quoque fit identica, ita ut duae priores classes nihil novi involvant. Neque tarnen hinc concludere licet, etiam in sequentibus ordinibus classes praecedentes praetermitti posse, siquidem in ordine sequente statim contrarium se manifestabit.

§ 88. Cum igitur hic ordo complectatur decem formulas integralium, earum valores per quattuor litteras \( A \), \( B \), \( P \) et \( Q \) ordine ita aspectui exponamus

1°. (1, 1) = \( \frac{A \cdot P}{B} \),

2°. (1, 2) = \( \frac{A \cdot Q}{B} \),

3°. (1, 3) = \( \frac{P}{B} \),

4°. (1, 4) = \( A \),

5°. (2, 2) = \( Q \),

6°. (2, 3) = \( B \),

7°. (2, 4) = \( \frac{B \cdot P}{B} \),

8°. (3, 3) = \( \frac{A \cdot B}{B} \),

9°. (3, 4) = \( \frac{A \cdot Q}{B} \),

10°. (4, 4) = \( \frac{A}{3 \cdot B} \).
SUPPLEMENTUM V.

§ 89. Cuius sit
\[ \frac{A}{B} = \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3}} = 2 \cos \frac{\pi}{3}; \]
tum vero
\[ \cos \frac{\pi}{3} = \frac{1 + \sqrt{3}}{2}, \text{ erit } \frac{A}{B} = \frac{1 + \sqrt{3}}{2}; \]
ideoque quantitas algebraica. Hinc igitur aliquot paria formularum integralium exhiberi poterunt, quae inter se teneant rationem algebraicam; erit enim
\[ \frac{(2, 1)}{(5, 2)} = \frac{1 + \sqrt{5}}{2}, \frac{(1, 2)}{(2, 3)} = \frac{1 + \sqrt{5}}{2}, \frac{(3, 4)}{(5, 3)} = \frac{1 + \sqrt{5}}{4}, \frac{(4, 3)}{(2, 4)} = \frac{1 + \sqrt{5}}{6}; \]
unde totidem egregia theoremeta condis possent, nisi ex his formulis manif esto elucendent.

Ordo IV.

quod \( n = 6 \) et formula
\[ (p, q) = \int \frac{x^{p-1} \, dx}{\sqrt[4]{(1 - x^6)^{6-q}}} = \int \frac{x^{q-1} \, dx}{\sqrt[6]{(1 - x^6)^{6-p}}}. \]

§ 90. Quoniam hic est \( n = 6 \), habebimus ante omnia
\[ (6, 1) = 1, (6, 2) = \frac{1}{2}, (6, 3) = \frac{1}{3}, (6, 4) = \frac{1}{4}; \]
formularum autem integralium in hoc ordine occurrentium numerus est 15, quae sunt
\[ 1. (1, 1), 2. (1, 2), 3. (1, 3), 4. (1, 4), 5. (1, 5), \]
\[ 6. (2, 2), 7. (2, 3), 8. (2, 4), 9. (2, 5), 10. (3, 3), \]
\[ 11. (3, 4), 12. (3, 5), 13. (4, 4), 14. (4, 5), 15. (5, 5); \]
inter quas reperiuntur tres circulares, quas singulari modo designamus, scilicet
\[ 1. (1, 5) = \frac{\pi}{6 \sin \frac{\pi}{6}} = \frac{\pi}{3} = A, \]
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20. \( (2, 4) = \frac{\pi}{6} \sin \frac{2\pi}{6} = \frac{\pi}{3} \sqrt{3} = B, \) et

30. \( (3, 3) = \frac{\pi}{6} \sin \frac{3\pi}{6} = \frac{\pi}{6} = C; \)

ita ut sit \( A = \frac{3}{2} C. \) Praeterea vero ambas formulæ, quae in ordine praecedente erant circulares, nunc vero sunt transcendentes, statuamus \( (1, 4) = P \) et \( (2, 3) = Q. \) His factis denominationibus evolvamus decem aequationes classis quartae, quae sunt

1°. \( B = P (5, 2), \)
2°. \( C = (3, 4) (4, 3), \)
3°. \( C = \frac{3}{2} Q (5, 3), \)
4°. \( B = (2, 4) (3, 4), \)
5°. \( B = (2, 2) (4, 4), \)
6°. \( B = (3, 5) (5, 4), \)
7°. \( A = (1, 1) (5, 2), \)
8°. \( A = (1, 2) (3, 5), \)
9°. \( A = (1, 3) (4, 5), \)
10°. \( A = 4 P (5, 5), \)

quas ita succinctius referre licet

\[
A = (1, 1) (5, 2) = 2 (1, 2) (3, 5) = 3 (1, 3) (4, 5) = 4 P (5, 5),
\]

\[
B = P (5, 2) = (2, 4) (3, 4) = 2 (2, 2) (4, 4) = 3 Q (4, 5),
\]

\[
C = (3, 4) (5, 2) = 2 Q (5, 3).
\]

Ecce ergo decem producta ex binis formulis integralibus, quorum singula quantitati circulari aequantur.

\[\text{\textsection 94.} \] Cum deinde sit \( \frac{\Delta}{\beta} = \sqrt{3} \) et \( \frac{\Delta}{\alpha} = 2, \) tum vero \( \frac{\beta}{\alpha} = \sqrt{3}, \) plura paria binarum formularum integralium exhibe-
beri possunt, quae inter se teneant rationem algebraicam; crit enim
\[
\frac{A}{B} = \sqrt[3]{3} = \frac{(1, 1)}{(1, 4)} = 2 \cdot \frac{(3, 5)}{(3, 4)} = \frac{(1, 3)}{(2, 5)} = 4 \cdot \frac{(5, 5)}{(5, 2)}
\]
\[
\frac{A}{C} = \frac{B}{C} = \frac{(1, 2)}{(1, 4)} = \frac{(1, 2)}{(2, 8)} = 3 \cdot \frac{(4, 5)}{(2, 5)}^2
\]
\[
\frac{B}{C} = \sqrt[3]{3} = \frac{(1, 4)}{(1, 3)} = 3 \cdot \frac{(4, 5)}{2 \cdot (3, 5)}.
\]
§ 92. Quodsi jam quinque formulas litteris A, B, C, P et Q designatas tanquam cognitas spectemus, videamus, quomodo reliquae formuleae per eam definire queant. Ac primo quidem percurrant decem aequationes classis quartae supra allatas, quarum prima dabit \((5, 2) = \frac{B}{P}\), tertia dat \((5, 3) = \frac{C}{2Q}\), sexta praebet \((5, 4) = \frac{B}{3Q}\), decima dat \((5, 5) = \frac{A}{P}\). Quodsi jam hos valores in reliquis surrogemus, secunda dabit \((3, 1) = \frac{C}{(1, 3)} = \frac{AQ}{B}\), septima praebet \((1, 1) = \frac{A}{(5, 2)} = \frac{AP}{B}\), octava dat \((1, 2) = \frac{A}{3(3, 5)} = \frac{AQ}{B}\), nona dat \((3, 1) = \frac{A}{3(4, 5)} = \frac{AQ}{B}\), quem valorem etiam secunda praebuat. Porro vero quarta dat \((3, 4) = \frac{B}{(2, 3)} = \frac{BC}{AQ}\). At vero ex aequatione quinta nullum valorem elicere possimus, quia neque formula \((2, 2)\) nec \((4, 4)\) etiamnun constat. Causa est quia duae reliquarum aequationum eandem determinationem produxerunt.

§ 93. Coacti igitur sumus, ad aequationes praeecedentium classium configurare, atque adeo ex prima classe
\[
(1, 2) \cdot (3, 1) = (1, 1) \cdot (2, 2)
\]
statim colligimus
\[
(2, 2) = \frac{(1, 2) \cdot (3, 1)}{(1, 1)} = \frac{AQQ}{CP},
\]
qui valor in quinta aequatione substitutus suppeditat postremam aequationem, nempe
\[
(4, 4) = \frac{B}{2(3, 2)} = \frac{BCP}{2AQQ}.
\]
Omnes igitur hos valores hic ordine referemus

1°. (1, 1) = \frac{A}{B} \cdot \frac{P}{B}.
2°. (1, 2) = \frac{A}{C} \cdot \frac{Q}{C}.
3°. (1, 3) = \frac{A}{D} \cdot \frac{Q}{D}.
4°. (1, 4) = P.
5°. (1, 5) = A.
6°. (2, 2) = \frac{A}{C} \cdot \frac{Q}{C}.
7°. (2, 3) = Q.
8°. (2, 4) = B.

9°. (2, 5) = \frac{B}{D}.
10°. (3, 3) = C.
11°. (3, 4) = \frac{B}{C} \cdot \frac{P}{C}.
12°. (3, 5) = \frac{B}{Q}.
13°. (4, 4) = \frac{B}{C} \cdot \frac{P}{C}.
14°. (4, 5) = \frac{B}{Q}.
15°. (5, 5) = \frac{A}{4P}.

§. 94. Cum autem in hoc ordine etiam aequationes tam classinis secundae quam tertiae valere debeat, videamus utrum valores inventi his classibus conveniant, an vero forte novam determinationem suppletent? Facta autem substitutione in tribus aequationibus secundae classis, ad identitatem pervenitur, quod idem quoque in aequationibus tertiae classis contingere debet, id quod evolventi mox patebit. Unde memorabile est omnes aequationes in quatuor primis classibus contentas, quarum numerus est 20, tantum decem determinationes in se complecti.

Ordo V.

quo \( n = 7 \) et formula

\[
(p, q) = \int \frac{x^{2-1} \, dx}{\sqrt{(1 - x^2)^{q-1}}} = \int \frac{x^{q-1} \, dx}{\sqrt{(1 - x^2)^{1-p}}}.
\]

§. 95. Quia hic \( n = 7 \), ante omnia habebimus valores absolutos \((7, 1) = 1, (7, 2) = \frac{1}{2}, (7, 3) = \frac{1}{3}, (7, 4) = \frac{1}{4}, (7, 5) = \frac{1}{5}\); deinde inter formulas integrales hujus ordinis imprimis
SUPPLEMENTUM V.

notari debent circulares, quas hoc modo designemus

\[ (4, 6) = \frac{\pi}{7 \sin \frac{\pi}{7}} = A , \]

\[ (2, 5) = \frac{\pi}{7 \sin \frac{2\pi}{7}} = B , \]

\[ (3, 4) = \frac{\pi}{7 \sin \frac{3\pi}{7}} = C . \]

Praeterea vero peculiaribus litteris notentur eae formulæ, quæ in ordine præcedenti erant circulares, hic autem valores transcenden-
tes sortiuntur, qui sint \( (4, 5) = P, (2, 4) = Q, \) et \( (3, 3) = R ; \) per hæs enim sex litteras videbimus omnes reliquas formulæ hujus
ordinis determinari posse.

§ 96. Quoniam supra non omnes aequationes quintae
classis expressimus, eas hic conjunctim exhibeamus, et ad nostrum
casum accommodemus

| I°. \( (1, 6)(7, 1) = (1, 1)(2, 6) \) | A = \( (4, 1)(2, 6) \),
| II°. \( (1, 6)(7, 2) = (1, 2)(3, 6) \) | A = \( 2(1, 2)(3, 6) \),
| III°. \( (1, 6)(7, 3) = (1, 3)(4, 6) \) | A = \( 3(1, 3)(4, 6) \),
| IV°. \( (1, 6)(7, 4) = (1, 4)(5, 6) \) | A = \( 4(1, 4)(5, 6) \),
| V°. \( (1, 6)(7, 5) = (1, 5)(6, 6) \) | A = \( 5 P (6, 6) \),
| VI°. \( (2, 5)(7, 1) = (2, 1)(3, 5) \) | B = \( 2(2, 1)(3, 5) \),
| VII°. \( (2, 5)(7, 2) = (2, 2)(4, 5) \) | B = \( 2(2, 2)(4, 5) \),
| VIII°. \( (2, 5)(7, 3) = (2, 3)(5, 5) \) | B = \( 3(2, 3)(5, 5) \),
| IX°. \( (2, 5)(7, 4) = (2, 4)(6, 5) \) | B = \( 4 Q (6, 5) \),
| X°. \( (3, 4)(7, 1) = (3, 1)(4, 4) \) | C = \( 3(3, 1)(4, 4) \),
| XI°. \( (3, 4)(7, 2) = (3, 2)(5, 4) \) | C = \( 2(3, 2)(5, 4) \),
| XII°. \( (3, 4)(7, 3) = (3, 3)(6, 4) \) | C = \( 3 R (6, 4) \),
| XIII°. \( (4, 3)(7, 1) = (4, 1)(6, 3) \) | C = \( 4(4, 1)(6, 3) \),
| XIV°. \( (4, 3)(7, 2) = (4, 2)(6, 3) \) | C = \( 2 Q (6, 3) \),
| XV°. \( (5, 2)(7, 1) = (5, 1)(6, 2) \) | B = \( 5 P (6, 2) \) |
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Hic igitur habemus quinque producta formulae A, aequalia, totidemque formulis B et C aequalia.


§. 98. Videamus igitur, quot harum formularum ex superioribus quindecim aequationibus determinare licet, ac primo quidem ex aequationibus V, IX, XII, XIV et XV, immediate deducuntur sequentes formularum (6, 6) = \frac{A}{B}, (6, 5) = \frac{B}{Q}, (6, 4) = \frac{C}{P}, (6, 3) = \frac{C}{Q}, (6, 2) = \frac{B}{P}. His jam inventis ex aequationibus I, II, III et IV, derivamus has formulas (1, 1) = \frac{A}{P}, (1, 2) = \frac{A}{C}, (1, 3) = \frac{A}{P}, (1, 4) = \frac{Q}{B}. Ex his vero valoribus per aequationes VI, X et XIII, colligimus (3, 5) = \frac{B}{C}, (4, 4) = \frac{C}{A}, et (5, 3) = \frac{B}{A}, ubi notasse jussit eundem valorem pro (5, 3) produisse ex aequationibus VI, et XIII. Ex reliquis autem aequationibus VII, VIII et IX, nihil concludere licet, unde istae quattuor formularum (2, 2), (2, 3), (5, 4), et (5, 5), nobis etiamnunc manent incognitae.

§. 99. Recurrere ergo coacti sumus ad aequationes praecedentium classium, quippe quae aequae ad nostrum ordinem pertinent atque aequationes classis quintae; hanc ob rem similis modo aequationes classis quartae hic opponamus et ad nostrum casum appliceimus.
SUPPLEMENTUM V.

\[ \begin{align*}
\text{I}^\circ. \quad (1,6) \quad & \equiv (4,1) \\ (2,5) \quad & \equiv (1,1) \quad B \\
\text{II}^\circ. \quad (1,5) \quad & \equiv (4,2) \\ (6,2) \quad & \equiv (1,2) (3,6) \\
\text{III}^\circ. \quad (1,5) \quad & \equiv (4,4) \\ (6,3) \quad & \equiv (1,3) (4,5) \\
\text{IV}^\circ. \quad (1,5) \quad & \equiv (4,4) \\ (6,4) \quad & \equiv (1,4) (5,5) \\
\text{V}^\circ. \quad (2,4) \quad & \equiv (2,4) \\ (6,1) \quad & \equiv (2,1) (3,4) \\
\text{VI}^\circ. \quad (2,4) \quad & \equiv (2,2) \\ (6,2) \quad & \equiv (2,2) (4,4) \\
\text{VII}^\circ. \quad (2,4) \quad & \equiv (2,8) \\ (6,3) \quad & \equiv (2,3) (5,4) \\
\text{VIII}^\circ. \quad (3,8) \quad & \equiv (3,4) \\ (6,1) \quad & \equiv (3,1) (4,3) \\
\text{IX}^\circ. \quad (3,3) \quad & \equiv (3,2) \\ (6,2) \quad & \equiv (3,2) (5,3) \\
\Xi^\circ. \quad (4,2) \quad & \equiv (4,1) \\ (6,4) \quad & \equiv (4,1) (5,2) \\
\end{align*} \]

\[ \text{§ 100. Ex aequationibus I, V, VIII, et } \Xi \text{ immediate concludimus has formulas } (4,1) = \frac{PA}{B}, (2,1) = \frac{QA}{C}, (3,1) = \frac{RA}{C}, (4,1) = \frac{AQ}{B}, \text{ quos autem valores jam ante adepti sumus. Secunda aequatio, si formulae jam inventae substituantur, praebet aequationem identicam. Ex tertia autem poterimus definire formulam (4,5), cuius valor hinc colligitur (4,5) = \frac{CP}{AQ} \cdot \text{ Simili modo ex IV elicimus (5,5) = } \frac{BCP}{AQR}. \text{ Porro ex aequatione VI concludimus fore (2,2) = } \frac{ABQR}{CQP} \cdot \text{ Deinde septima aequatio dat (2,3) = } \frac{AQR}{CQP}. \text{ Nona vero aequatio etiam praebet (3,2) = } \frac{AQR}{CQP}. \text{ Sicque omnes quindecim formularum incognitae determinavimus per sex litteras cognitas A, B, C, P, Q et R.} \]

\[ \text{§ 101. Valores igitur omnium formularum hujus ordinis hic aspectui conjunctim exponamus} \]

\[ (1,6) \]
§. 102. Quoniam autem aequationes primae, secundae ac tertiae classis etiam in hoc ordine valent, si in iis valores hic inventos substituimus, perpetuo in aequationes identicas incidimus. Ita cum aequatio primae classis sit \((1, 2) (3, 4) = (1, 1)\) \((2, 2)\), facta substitutione reperitur \((1, 2) (3, 4) = \frac{AQR}{CC}\); at vero \((1, 1) (2, 2)\) fit \(= \frac{AQR}{CC}\), haecque identitas etiam deprehendetur, in tribus aequationibus secundae classis, atque etiam in sex aequationibus tertiae classis, quemadmodum calculum instituenti mox patebit.

§. 103. Simili modo haud difficile erit hanc investigationem ad ordinates superiores extendere, neque tamen legem observare licet, secundum quam determinationes singularum formulæ in eaqueque ordinis progrediuntur. Interim tamen observasse juvabit, in ordine sequente sexto, ubi \(n = 8\) et formæae occurrunt 28, eas omnes primo per quatuor formulas circulares \((1, 7) = A, (2, 6) = B, (3, 5) = C, (4, 4) = D\), praeterea vero per has tres transcendentes \((1, 6) = P, (2, 5) = Q, (3, 4) = R\), determinari posse. Cum igitur quovis ordine determinatio singularum formulae, praeter formulas circulares, quae utique pro cognitis haberi possunt, etiam aliquot formulas transcendentes postulat, si saltem valores harum formularum vero proxime cognoscere voluerimus, methodus adhuc Vol. IV.
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desideratur, istos valores proxime, veluti in fractionibus decimallis, definendi. Talem igitur methodum hic coronidis loco subjugemus.

**Problema.**

*Proposita formula integrali cujusque ordinis*

\[ S = \int \frac{x^{p-1} \, dx}{\sqrt[1-x^n]{1-x^n}} \]

*a termino \( x = 0 \) usque ad \( x = 1 \) extendenda, investigare series convergentem, quae istum valorem \( S \) exprimat.*

**Solutio.**

§ 104. Cum sit

\[ \frac{1}{\sqrt[1-x^n]{1-x^n}} = (1-x^n)^{(n-q)} \]

facta evolutione hujus potestatis binomii more solito, reperietur

\[ \frac{1}{\sqrt[1-x^n]{1-x^n}} = 1 + \frac{n-q}{n} x^n + \frac{n-q}{n} \frac{2n-q}{2n} x^{2n} \]

\[ + \frac{n-q}{n} \frac{2n-q}{2n} \frac{3n-q}{3n} x^{3n} + \text{ etc.} \]

Si haec series ducatur in \( x^{p-1} \, dx \) et integretur, probitet

\[ S = \frac{x^p}{p} + \frac{n-q}{n} \frac{x^{2n+p}}{2n+p} + \frac{n-q}{n} \frac{2n-q}{2n} \frac{x^{2n+p}}{2n+p} \]

\[ + \frac{n-q}{n} \frac{2n-q}{2n} \frac{3n-q}{3n} \frac{x^{3n+p}}{3n+p} + \text{ etc.} \]

quae series jam evanescit posito \( x = 0 \); unde si ponamus \( x = 1 \),

valor quaesitus nostrae formulae fiet
\[ S = \frac{x}{p} + \frac{n-q}{n} \cdot \frac{1}{n+p} + \frac{2n-q}{2n} \cdot \frac{1}{2n+p} \\
+ \frac{n-q}{2n} \cdot \frac{3n-q}{3n} \cdot \frac{1}{3n+p} + \text{etc.} \]

\[ \text{§ 105. Verum ista series, quicunque numeri pro litteris } n, p \text{ et } q \text{ accipientur, nimirum lente convergit, quam ut ex ea valores ipsius } S \text{ saltem ad tres quattuor figuras decimales satis exacte definiri quacant; quamobrem aliam evolutionem instituir conveniet, dum scilicet valorem quae situm in duas partes resolvimus. Statuimus igitur} \]
\[ \int_{x_0}^{x_f} \frac{x^{p-1} \, dx}{\sqrt{(1-x^n)^{n-q}}} \left[ \begin{array}{c} \text{ab } x = 0 \\ \text{ad } x = \frac{1}{2} \end{array} \right] = P \text{ et} \]
\[ \int_{x_0}^{x_f} \frac{x^{p-1} \, dx}{\sqrt{(1-x^n)^{n-q}}} \left[ \begin{array}{c} \text{ab } x = \frac{1}{2} \\ \text{ad } x = 1 \end{array} \right] = Q, \]

atque evidens est fore \( S = P + Q \). Nunc autem tam pro \( P \) quam pro \( Q \) haud difficuler series satis convergentes exhibere poterunt.

\[ \text{§ 106. Quod primum ad valorem } P \text{ attinet, cum ex valore generali, quem supra pro } S \text{ invenimus, facile derivabimus ponendo } x^n = \frac{1}{2}, \text{ ita ut sit } x = \sqrt[2n]{\frac{1}{2}} \text{ et } x^p = \frac{1}{\sqrt{2^n}}, \text{ quo facto pro } P \text{ obtinebimus hanc seriem} \]
\[ P = \frac{1}{\sqrt{2^n}} \left( \frac{1}{p} + \frac{n-q}{2n} \cdot \frac{1}{n+p} + \frac{n-q}{2n} \cdot \frac{2n-q}{4n} \cdot \frac{1}{2n+p} \right. \]
\[ + \frac{n-q}{2n} \cdot \frac{2n-q}{4n} \cdot \frac{3n-q}{6n} \cdot \frac{1}{3n+p} + \text{etc.} \]

In qua serie singuli termini plus quam in ratione dupla decres-
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cunt; ita ut verbi gratia terminus decimus jam multo minor futurus sit quam \( \frac{1}{10^4} \), unde si ad partes millionesimas certi esse velimus, sufficeret calculum nequidem ad vigesimum usque terminum extendere.

§. 107. Cum deinde posuerimus.

\[
Q = \int_{n}^{y} \frac{x^{p-1}}{\sqrt[(p-1)]{(1-x^n)^{n-q}}} \, dx \quad \left[ \text{ab } x^n = \frac{1}{2} \right] \quad \text{ad } x = 1
\]

statuamus \( 1 - x^n = y^n \), ut sit \( Q = \int_{y^n}^{n} \frac{x^{p-1}}{y^{n-q}} \, dx \), tum vero erit \( x^n = 1 - y^n \), ideoque \( x^p = \sqrt[n]{(1 - y^n)^p} \), unde differentiando colligitur:

\[
x^{p-1} \, dx = - y^{n-1} \, dy \left( 1 - y^n \right)^{\frac{p-n}{n}},
\]

quod valor substituto erit

\[
Q = - \int y^{n-1} \, dy \left( 1 - y^n \right)^{\frac{p-n}{n}} \quad \left[ \text{ab } y^n = \frac{1}{2} \right] \quad \text{ad } y = 0.
\]

Quando enim fit \( x^n = \frac{1}{2} \), tum etiam erit \( y^n = \frac{1}{2} \), at facto \( x = 1 \), manifesto fit \( y = 0 \); quare si terminos integrationis permutemus, etiam signum ipsius formulae immutari debet, sicque fiet

\[
Q = \int \beta^{q-1} \, dy \left( 1 - y^n \right)^{\frac{p-q}{n}} \quad \left[ \text{ab } y = 0 \right] \quad \text{ad } y^n = \frac{1}{2}.
\]

§. 108. Haec autem formula pro \( Q \) inventa omnino similis est illi, quam pro \( P \) invenimus, hoc tantum discrimine, quod litterae \( p \) et \( q \) inter se sunt permutatae; quocircum, si integrata per seriem institutur, proveniet sequens
\[ Q = \frac{1}{\sqrt{2^p}} \left( \frac{1}{q} + \frac{n-p}{2n} + \frac{n-p}{2n} + \frac{2n-p}{4n} + \frac{1}{2n+q} \right) \]

\[ + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+q} + \text{etc.} \]

quae series acque converget, ac praecedens pro P inventa. His autem duabus seriebus ad calculus revocatis semper erit valor quae situs \( S = P + Q \).

\textbf{Corollarium 1.}

\( \S \, 109. \) Iste calculus plurimum contrahetur iis casibus, quibus \( p = q \), tum enim fit \( P = Q \), hisque casibus, quibus

\[ S = \int_{0}^{x} \frac{x^{p-1}}{\sqrt{(1-x^n)^{n-p}}} \, dx \]

valor istius formulae ab \( x = 0 \) ad \( x = 1 \) extensae erit

\[ S = \frac{2}{\sqrt{2^p}} \left( \frac{1}{p} + \frac{n-p}{2n} + \frac{1}{n+p} + \frac{n-p}{2n} + \frac{2n-p}{4n} + \frac{1}{2n+p} \right) \]

\[ + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+p} + \text{etc.} \]

\textbf{Corollarium 2.}

\( \S \, 110. \) Quoniam igitur in singulis ordinebus nonnullae hujusmodi formularum \( (p, p) \) occurrunt, statim atque valores aliquot hujusmodi formularum fuerint ad calculus decimalem revocati, quoniam formularum circulares per se sunt notae, ex iis valores omnium reliquarum formularum ejusdem ordinis assignare licebit.
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Exemplum.

§. 111. Proposita sit formula ordinis primi, ubi \( p = q = 2 \)
et \( S = \int \frac{x \, dx}{\sqrt{1 - x^3}} \). Series igitur pro \( S \) inventa erit

\[
S = \sqrt{2} \left( 1 + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{1}{8} + \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{1}{16} + \frac{1}{6} \cdot \frac{1}{18} \cdot \frac{1}{20} \cdot \frac{1}{32} + \frac{1}{6} \cdot \frac{1}{24} \cdot \frac{1}{36} + \frac{1}{6} \cdot \frac{1}{28} \cdot \frac{1}{40} \cdot \frac{1}{48} + etc. \right).
\]

Subducta autem calculo reperitur

\( S = 0, 54325 \times \sqrt[3]{2} = 0, 68445 \),

qui ergo est valor formulæ (2, 2) in ordine \( \frac{1}{2} \) §. 22. ubi invenimus (2, 2) = \( \frac{A}{P} \), ita ut jam sit \( P = \frac{A}{(2, 2)} \). Est vero

\( A = \frac{2\pi}{3\sqrt{3}} = 1, 20918 \), hinc erit \( P = 2, 22582 = (1, 1) \): unde in fractionibus decimalibus ternae formulæ ordinis primi erunt

\( (1, 1) = 2, 22582, (1, 2) = 1, 20918, (2, 2) = 0, 68445 \).

Hocque modo etiam omnes formulas sequentium ordinum evolvere licet.

3) Additamentum ad Dissertationem praecedentem, de valoribus formulæ integralis

\[
\int \frac{x^{p-1} \, dx}{\sqrt[3]{1 - x^3}},
\]


§. 112. Si methodum in praecedente dissertatione traditam ad altiores ordines quam \( n = 7 \) transferre vellemus, ob
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ingenem aequationum considerandarum numerum labor fieret nimirum molestus. Quoniam autem vidimus, non omnes istas aequationes concurrense ad valores singularum formularum determinandos, opus non mediocriter sublevabitur, si quovis casu tantum aequationes in computum duceamus, quae immediate ad determinationes formularum perducent, quemadmodum hic pro casu \( n = 10 \) sum ostensurus.

Determinatio

harum formularum pro casu \( n = 10 \), ubi formula

\[
(p, q) = \int \frac{x^p - 1}{\sqrt{(1 - x^2)^{10 - q}}} \, dx = \int \frac{x^q - 1}{\sqrt{(1 - x^1)^{10 - p}}}.
\]

§ 113. Hoc casu ergo formae valorem absolutum recipientes sunt \((10, 1) = 4\), \((10, 2) = \frac{1}{2}\), \((10, 3) = \frac{1}{3}\) et in genere \((10, a) = \frac{1}{a}\). Deinde omnes formae, in quibus est \(p + q = 10\), a circulo pendent, ideoque pro cognitis haberi possunt, quas ergo propria litteris designemus

\[
\begin{align*}
(1, 0) &= \frac{\pi}{10 \sin \frac{\pi}{10}} = A, \\
(2, 2) &= \frac{\pi}{10 \sin \frac{\pi}{5}} = B, \\
(3, 1) &= \frac{\pi}{10 \sin \frac{\pi}{3}} = C, \\
(4, 0) &= \frac{\pi}{10 \sin \frac{\pi}{5}} = D, \\
(5, 5) &= \frac{\pi}{10 \sin \frac{\pi}{5}} = E, \\
(6, 4) &= \frac{\pi}{10 \sin \frac{\pi}{6}} = D, \\
(7, 3) &= \frac{\pi}{10 \sin \frac{\pi}{3}} = C, \\
(8, 2) &= \frac{\pi}{10 \sin \frac{\pi}{6}} = B, \\
(9, 4) &= \frac{\pi}{10 \sin \frac{\pi}{6}} = A.
\end{align*}
\]
Supplementum V.

§. 114. Per hæs autem formulas circulares reliquas in forma generali contentas nequitiam determinare lícet; sed in super aliquot formulas transcendentes in subsidium vocari oportet, ex quibus cum circularibus illis conjunctis reliquarum omnium valores assignare lícet. Nostro autem casu, quo \( n = 10 \), sequentes formulas tanquam cognitas spectari convéniet, quae in ordine præcedentis, ubi \( n = 9 \), erant circulares, nunc autem in ordinem transcendentium transeunt. Eas igitur sequenti modo designemus:

\[
\begin{align*}
(1, 8) & = P, \quad (2, 7) = Q, \quad (3, 6) = R, \quad (4, 5) = S, \\
(5, 4) & = S, \quad (6, 3) = R, \quad (7, 2) = Q, \quad (8, 1) = P.
\end{align*}
\]

Scilicet si valores harum litterarum quoque tanquam cognitos spectemus, per eos cum circularibus junctos reliquas formulas omnes in hoc ordine contentas determinare poterimus. Cum igitur numerus omnium formularum integralium in hoc ordine \( n = 10 \) contentarum sit 45, ex illis autem novem ut cognitae spectentur, reliquae 36 per hæs litteras majusculas determinari debebunt.

§. 115. Istas autem determinationes ex aequatione generali supra demonstrata peti oportet, quae haec forma continetur:

\[
(a, b) \cdot (a + b, c) = (a, c) \cdot (a + c, b);
\]

ubi assumere lícet, semper esse \( b > c \), quoniam, si foret \( c = b \), aequatio foret identica. Primo igitur ut hinc aequationes, quae immediate determinationes praebent, nanciscamur, sumamus \( a + b = 10 \), ut sit \( (10, c) = \frac{c}{2} \); tum vero capiatur \( c = b - 1 \), quo facto pro \( a \) ordine scribendo numeros 1, 2, 3, etc. sequentes prodibunt determinationes:

\[
\begin{align*}
(1, 9) \cdot (10, 8) & = (1, 8) \cdot (9, 9), \text{ sive } \frac{1}{3} A = P (9, 9), \text{ ergo } \\
(9, 9) & = \frac{1}{3} P, \quad (9, 9) \cdot (9, 8) = \frac{1}{3} P. \\
(2, 8) \cdot (10, 7) & = (2, 7) \cdot (9, 8), \text{ sive } \frac{1}{3} B = Q (9, 8), \text{ ergo } \\
(9, 8) & = \frac{1}{3} Q.
\end{align*}
\]
(3, 7) (10, 6) = (3, 6) (9, 7), sive \( \frac{2}{3} \ C = R (9, 7) \), ergo 
(9, 7) = \( \frac{C}{6R} \).

(4, 6) (10, 5) = (4, 5) (9, 6), sive \( \frac{1}{3} \ D = S (9, 6) \), ergo 
(9, 6) = \( \frac{B}{5S} \).

(5, 5) (10, 4) = (5, 4) (9, 5), sive \( \frac{1}{3} \ E = S (9, 5) \), ergo 
(9, 5) = \( \frac{E}{4S} \).

(6, 4) (10, 3) = (6, 3) (9, 4), sive \( \frac{1}{3} \ D = R (9, 4) \), ergo 
(9, 4) = \( \frac{D}{3R} \).

(7, 3) (10, 2) = (7, 2) (9, 3), sive \( \frac{1}{3} \ C = Q (9, 3) \), ergo 
(9, 3) = \( \frac{C}{2Q} \).

(8, 2) (10, 1) = (8, 1) (9, 2), sive \( \frac{1}{2} \ B = P (9, 2) \), ergo 
(9, 2) = \( \frac{B}{P} \).

§. 116. Ex formulis igitur incognitis illis numero 36 jam octo determinavimus, quae nobis viam sternent ad novas determinationes, quas primo derivabimus ex aequatione generali summendo \( a = 1, b = 9 \), et pro c scribendo ordine numeros 1, 2, 3 . . . . . 8, unde calculus ita se habebit

\[
\begin{align*}
(1, 9) (10, 1) &= (1, 1) (2, 9) & A &= (1, 1) \frac{B}{P}, \text{ ergo } (1, 1) &= \frac{AP}{B} \\
(1, 9) (10, 2) &= (1, 2) (3, 9) & \frac{1}{2} A &= (1, 2) \frac{C}{Q}, \text{ ergo } (1, 2) &= \frac{AQ}{C} \\
(1, 9) (10, 3) &= (1, 3) (4, 9) & \frac{1}{3} A &= (1, 3) \frac{D}{R}, \text{ ergo } (1, 3) &= \frac{AR}{D} \\
(1, 9) (10, 4) &= (1, 4) (5, 9) & \frac{4}{4} A &= (1, 4) \frac{E}{S}, \text{ ergo } (1, 4) &= \frac{AS}{E} \\
(1, 9) (10, 5) &= (1, 5) (6, 9) & \frac{5}{5} A &= (1, 5) \frac{F}{S}, \text{ ergo } (1, 5) &= \frac{AF}{F} \\
(1, 9) (10, 6) &= (1, 6) (7, 6) & \frac{6}{6} A &= (1, 6) \frac{G}{R}, \text{ ergo } (1, 6) &= \frac{AG}{R} \\
(1, 9) (10, 7) &= (1, 7) (8, 9) & \frac{7}{7} A &= (1, 7) \frac{H}{Q}, \text{ ergo } (1, 7) &= \frac{AH}{Q} \\
(1, 9) (10, 8) &= (1, 8) (9, 9) & \frac{8}{8} A &= (1, 8) \frac{I}{P}, \text{ ergo } (1, 8) &= \frac{AI}{P}
\end{align*}
\]

Hocque modo septem novas determinationes sumus adepti.
§. 117. His autem inventis consideremus aequationes ex valoribus $a = 1$, $b = 8$, $c = 1$, $2$, ... $7$ ortas, eritque

$(1, 8) \cdot (9, 1) = (4, 1) \cdot (2, 8)$
$(1, 8) \cdot (9, 2) = (4, 2) \cdot (3, 8)$
$(1, 8) \cdot (9, 3) = (4, 3) \cdot (4, 8)$
$(1, 8) \cdot (9, 4) = (4, 4) \cdot (5, 8)$
$(1, 8) \cdot (9, 5) = (4, 5) \cdot (6, 8)$
$(1, 8) \cdot (9, 6) = (4, 6) \cdot (7, 8)$
$(1, 8) \cdot (9, 7) = (4, 7) \cdot (8, 8)$

$AP = (1, 4) \cdot B$  
$BC = (3, 8) \cdot \frac{AQ}{C}$  
$(3, 8) = \frac{BC}{AQ}$  
$(4, 8) = \frac{CDP}{AQ}$  
$(5, 8) = \frac{ABR}{CDP}$  
$(6, 8) = \frac{ABR}{CDP}$  
$(7, 8) = \frac{ABR}{CDP}$  
$(8, 8) = \frac{ABR}{CDP}$

§. 118. Novas determinationes reperiemus ponendo $a = 1$, $b = 7$, $c = 3$, $4$, $5$, $6$; hinc enim nanciscimus sequentes determinationes

$(1, 7) \cdot (8, 3) = (4, 3) \cdot (4, 7)$
$(1, 7) \cdot (8, 4) = (4, 4) \cdot (5, 7)$
$(1, 7) \cdot (8, 5) = (4, 5) \cdot (6, 7)$
$(1, 7) \cdot (8, 6) = (4, 6) \cdot (7, 7)$

$C = (4, 7) \cdot \frac{AR}{D}$  
$(4, 7) = \frac{CDP}{AR}$  
$(5, 7) = \frac{CDP}{AR}$  
$(6, 7) = \frac{CDP}{AR}$  
$(7, 7) = \frac{CDP}{AR}$

§. 119. Sumamus nunc $a = 1$, $b = 6$, $c = 4$, $5$, eritque

$(1, 6) \cdot (7, 4) = (4, 4) \cdot (5, 6)$
$(1, 6) \cdot (7, 5) = (4, 5) \cdot (6, 6)$

Hactenus igitur omnes formulas $(p, q)$ determinavimus, in quibus $p + q > 10$. Ex reliquis autem, ubi $p + q > 9$, jam nactis sumus istas.
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§ 120. Pro his inveniendis sumamus \( a = 1 \) et \( c = 1 \), pro \( b \) autem ordine capiamus numeros 2, 3, etc. atque consequemur has aequationes

\[
\begin{align*}
(1, 2) &\quad (3, 1) = (1, 1) \cdot (2, 2) & \frac{AAQR}{CD} &= (2, 2) \cdot \frac{AP}{B} & (2, 2) &= \frac{ABQR}{CDP} \\
(1, 3) &\quad (4, 1) = (1, 1) \cdot (2, 3) & \frac{AARS}{DE} &= (2, 3) \cdot \frac{AP}{B} & (2, 3) &= \frac{ABRS}{DEP} \\
(1, 4) &\quad (5, 1) = (1, 1) \cdot (2, 4) & \frac{AASS}{DE} &= (2, 4) \cdot \frac{AP}{B} & (2, 4) &= \frac{ABSS}{DEP} \\
(1, 5) &\quad (6, 1) = (1, 1) \cdot (2, 5) & \frac{AAQR}{BC} &= (2, 5) \cdot \frac{AP}{B} & (2, 5) &= \frac{ABRS}{CDP} \\
(1, 6) &\quad (7, 1) = (1, 1) \cdot (2, 6) & \frac{AAQR}{BC} &= (2, 6) \cdot \frac{AP}{B} & (2, 6) &= \frac{ABQR}{BCP} \\
\end{align*}
\]

sicque etiamnunc determinandae restant formae \((3, 3), (3, 4), (3, 5)\) et \((4, 4)\).

§ 121. Pro his sumatur \( a = 1, c = 2, \) et \( b = 3, 4, 5, \) etc. tum enim prodibunt hae aequationes

\[
\begin{align*}
(1, 3) &\quad (4, 2) = (1, 2) \cdot (3, 3) & \frac{AABRSS}{DDEF} &= (3, 3) \cdot \frac{AQ}{C} & (3, 3) &= \frac{ABCRSS}{DDEPF} \\
(1, 4) &\quad (5, 2) = (1, 2) \cdot (3, 4) & \frac{AABRSS}{DDEF} &= (3, 4) \cdot \frac{AQ}{C} & (3, 4) &= \frac{ABSS}{DDEP} \\
(1, 5) &\quad (6, 2) = (1, 2) \cdot (3, 5) & \frac{AAORS}{CDEF} &= (3, 5) \cdot \frac{AQ}{C} & (3, 5) &= \frac{ABSS}{DDEP} \\
\end{align*}
\]

Unica ergo formula restat determinanda, scilicet \((4, 4)\), quae ex hae aequatione \((1, 4) \cdot (5, 3) = (1, 3) \cdot (4, 4)\) definietur; etiam \(\frac{AABRRSS}{DDEF} = (4, 4) \cdot \frac{AR}{D}\), ideoque \((4, 4) = \frac{ASS}{EP}\).

42°
§. 122. Ut nunc omnes has determinationes simul aspectui exponamus, quoniam in hoc ordine $n = 10$ omnino 45 formulae integrales occurrunt, si ex his ut cognitae spectentur novem sequentes


reliquae triginta sex ex his sequenti modo determinabantur.

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AD TOM. I. CAP. VIII.

§. 123. Eadem methodo, qua hic usi sumus pro casu

\[ n = 10 \], haud difficile erit ordines altiores evolvere; neque tamen

hinc adhuc elucet, quanam lege omnes determinationes progre-
diantur, quandoquidem valores certarum formularum continuo ma-
gis evadunt complicati. Ceterum valores, quos hic invenimus,
onnibus aequationibus in forma generali

\[ (a, b) (a + b, c) = (a, c) (a + c, b) \]

contentis satisfacere deprehenduntur, ita ut perpetuo aequatio
identica resultet, neque idcirco inde utra nova relatio inter litte-
ras nostras majusculas deduci queat. Tandem probe hic notasse
juvabit, quod in omnibus ordinibus, praeter formulas a circulo
pendentes, commodissimae eae formulae, quae in ordine pro-
ximae praecedente erant circulares, hic etiam tanquam cognitae
acipi quaeant, quippe quibus determinationes omnes optimo
successu perfici possunt.

Methodus generalis determinandi valores
formulae

\[ (p, q) = \int_{\frac{p}{q}}^{\frac{p+q}{n}} \frac{dx}{\sqrt{(1 - x^n)^{n-q}}} = \int_{\frac{p}{q}}^{\frac{p+q}{n}} \frac{dx}{\sqrt{(1 - x^n)^{n-p}}} \]

a termino \( x = 0 \) usque ad \( x = 1 \) extensa: ubi praeter
formulas circulum involventes, in quibus est \( p+q = n \),
etiam illae pro cognitis accipiuntur, in quibus est
\( p+q = n-1 \).

I. Cum aequatio generalis, unde omnes haec determina-
tiones sunt petendae, sit

\[ (a, b) (a + b, c) = (a, c) (a + c, b) \],
SUMPLEMENTUM V.

sumatur primo \( a = n - a \), \( b = a \), et \( c = a - 1 \), eritque aequatio

\[
(n - a, a) (n, a - 1) = (n - a, a - 1)(n - 1, a),
\]

ubi est \( (n, a - 1) = \frac{n}{a - 1} \). In primo autem factore, ob \( p = n - a \)
et \( q = a \), est \( p + q = n \), ideoque datur. In tertio porro factore, ubi \( p = n - a \)et \( q = a - 1 \), est \( p + q = n - 1 \), ideoque pariter datur. Hinc ergo colligimus

\[
(n - 1, a) = \frac{1}{n - a} \frac{(n - a, a)}{(n - a, a - 1)},
\]

ubi esse debet \( n > 1 \), ita ut pro \( a \) accipi queant omnès numeri a 2 usque ad \( n - 1 \); at vero casu \( n = 1 \) valor formulae per se est notus.

II. In aequatione generali jam sumatur \( a = \beta \), \( b = n - \beta - 1 \), et \( c = 1 \), eritque nostra aequatio

\[
(\beta, n - \beta - 1)(n - 1, 1) = (\beta, 1)(\beta + 1, n - \beta - 1),
\]

ex qua aequatione colligitur

\[
(\beta, 1) = \frac{(\beta, n - \beta - 1)(n - 1, 1)}{(\beta + 1, n - \beta - 1)},
\]

ubi esse debet \( \beta < n - 1 \), ita ut hinc omnes formulae \( (\beta, 1) \) definiatur, a valore \( \beta = 1 \) usque ad \( \beta = n - 1 \), quo posterior caso formula \( (n - 1, 1) \) per se cognoscitur.

III. Ut hinc etiam alias formas eliciamus, sumamus \( a = 1 \), \( b = n - 2 \), \( c = \gamma \), ut oriatur haec aequatio

\[
(1, n - 2)(n - 1, \gamma) = (1, \gamma)(1 + \gamma, n - 2),
\]

ubi primus factor ac tertius dantur per \( N_0 \). II. secundus vero per \( N_0 \). I. unde quartus derivatur, scilicet

\[
(1 + \gamma, n - 2) = \frac{(1, n - 2)(n - 1, \gamma)}{(1, \gamma)}
\]

ubi valores ipsius \( 1 + \gamma \) a 2 usque ad \( n - 2 \) augeri possunt.
Cum igitur per N°. I. sit

\[(n - 1, \gamma) = \frac{(n - 1, \gamma - 1)}{(n - 1, \gamma - 1)}\]

tum vero per N°. II. fit

\[(\gamma, 1) = \frac{(\gamma, n - \gamma - 1)}{(\gamma + 1, n - \gamma - 1)}\]

hinc valoribus substitutis fiet

\[(n - 2, 1 + \gamma) = \frac{(n - 2, \gamma)}{(n - 3, \gamma - 1)} \cdot \frac{(1, n - 2) (n - 1, \gamma - 1)}{(n - 1, \gamma - 1)}\]

IV. Sumamus nunc \[a = 1, \ b = n - 3, \ c = \delta\], pro-
dibitque haec equatio

\[\left(1, n - 3\right), \left(n - 2, \delta\right) = \left(1, \delta\right) (1 + \delta, n - 3)\]

unde colligitur

\[\left(n - 3, 1 + \delta\right) = \frac{(n - 3, \delta)}{(1, \delta)}\]

ubi ergo \[1 + \delta\] contineat numeros \[2, \delta, 4, \ldots, n - 3\],
ita ut hinc excludatur \[n = 3, \delta\], quae autem per N°. I. datur.
At si valores ante reperti substituantur, fiet

\[\left(n - 3, 1 + \delta\right) = \frac{(n - 3, \delta)}{(n - 3, \delta)} \cdot \frac{(n - 2, \delta) (n - 2, \delta - 1)}{(n - 2, \delta) (n - 2, \delta - 1)}\]

unde patet esse dèbere \[\delta > 2\], codemque modo prò præce-
dente formula \[\gamma > 1\], ita ut hinc excludatur casus \(n - 3, 4\),
\(n - 3, 2\), quorum quidem prior per N°. I. datur, alter vero
per se.

V. Statuamus nunc \[a = 1, \ b = n - 4\] et \[c = \varepsilon\], pro-
dibitque haec equatio

\[\left(1, n - 4\right), \left(n - 3, \varepsilon\right) = \left(1, \varepsilon\right) (1 + \varepsilon, n - 4)\]

unde concluditur

\[\left(n - 4, 1 + \varepsilon\right) = \frac{(n - 4, \varepsilon)}{(1, \varepsilon)}\]

ubi si loco \(n - 3, \varepsilon\) valor ante inventus substituetur, factor
SUPPLEMENTUM V.

absolutus ingredetur $\frac{1}{e-3}$, ita ut esse debeat $e > 3$, ideoque $1 + e > 4$, unde hic excluduntur casus $(n - 4, 1), (n - 4, 2), (n - 4, 3)$, quorum quidem primus ex $N^0$. II. tertius autem per se datur, medius vero revera manet incognitus.

VI. Statuamus porro $a = 1$, $b = n - 5$, $c = z$, et aequatio erit

$$(1, n - 5) (n - 4, z) = (1, z) (1 + z, n - 5),$$

unde fit

$$(n - 5, 1 + z) = (n - 5, 1) (n - 5, z),$$

ubi ob formulam $(n - 4, z)$ debet esse $z > 4$, ideoque $1 + z > 5$, unde hinc excluduntur casus $(n - 5, 1), (n - 5, 2), (n - 5, 3)$, $(n - 5, 4)$, quorum quidem primus ex $N^0$. II. constat, quartus vero per se datur, ita ut hic occurrant duo casus etiam nunc incogniti $(n - 5, 2)$ et $(n - 5, 3)$.

VII. Simili modo si ulteriorius sumamus $a = 4$, $b = n - 6$ et $c = \eta$, prohibit

$$(n - 6, 1 + \eta) = (n - 6, 1) (n - 5, \eta),$$

ubi revera occurrunt tres sequentes casus $(n - 6, 2), (n - 6, 3), (n - 6, 4)$, qui adhuc manent incogniti, atque hoc modo progressi licebit, quousque necesse fuerit; unde patet numerum casuum incognitorum continuo augeri, ita ut terminorum $p$ et $q$ alter futurus sit vel 2, vel 3, vel 4, etc. qui igitur casus adhuc definiti restant.

VIII. Sumamus nunc primo $a = 1$, $b = \emptyset$, $c = 1$, ut aequatio nostra fiat

$$(1, \emptyset) (1 + \emptyset, 1) = (1, 1) (2, \emptyset),$$
unde concludimus

\[(2, b) = \frac{(1, b)(1 + 4, 2)}{(1, 2)}\]

quae formula jam omnes casus exclusos suppeditat, in quibus alter terminus erat 2.

IX. Deinde sumamus \(a = 2, b = \kappa\) et \(c = 1\), ut aequatio prodeat \((2, \kappa)(2 + \kappa, 1) = (2, 1)(3, \kappa)\), unde fit

\[(3, \kappa) = \frac{(2, \kappa)(2 + \kappa, 1)}{(3, 1)}\]

ubi \(\kappa\) per praecedentem \(N\)num detur, nunc etiam ii casus innotescunt, ubi alter terminus erat 3.

X. Sumatur porro \(a = 3, b = \kappa, c = 1\), et quae \((3, \kappa)\)

\[(3 + \kappa, 1) = (3, 1)(4, \kappa),\] unde fit

\[(4, \kappa) = \frac{(3, \kappa)(3 + \kappa, 1)}{(3, 1)}\]

unde igitur ii casus eliciumtur, ubi alter terminus erat 4. Eodem modo pro reliquis proceeditur; sicque omnes plane casus in formula proposita contenti plene sunt determinati.

4) De valoribus integralium a termino variabilis \(x = 0\) usque ad \(x = \infty\) extensorum. M. S. Academiae exhib. d. 30 Aprilis 1781.

§. 124. Talium formularum, quae a termino \(x = 0\) usque ad terminum \(x = \infty\) extensae finitum sortiuntur valorem, simplicissima est circularis \(\int \frac{dx}{1 + x^2}\), cujus valor est \(\frac{\pi}{2}\) denotante \(\pi\) peripheriam pro diametro \(= 1\). Deinde etiam modo prorsus singulari inveni esse

\[
\int \frac{x^{m-1}}{(1 + x^2)^n} dx \left[ \begin{array}{c} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\pi}{n \sin \frac{m \pi}{n}}.
\]
SUPPLEMENTUM V.

Praeterea vero hoc modo plures alias formulas hujus generis expedivi, in quorum differentiaaliam non solum functiones algebraicae ipsius $x$ sed etiam $lx$ ingrediatur.

§. 126. Obtulerunt se mihi autem quondam aliae hujusmodi formaeae etiam functiones transcendentes involventes, quorum valores desiderati omnes methodos adhuc cognitas respuere videantur. Quaesiveram scilicet eam lineam curvam in qua radius osculi ubique reciprocro esse proportionalis arcui curvae, ita ut posito arcu $s$ et radio osculi $r$, esset $rs = aa$. Hinc enim haud difficile est, figuram curvae libro quasi manus duci describere, quandoquidem ea talem habere debet figuram. Inuito nimirum curvae in $A$ constito inde curva continuo magis incurvabitur et tandem post infinitas spiras in certum punctum $O$ glomerabitur, quod polum hujus curvae appellare licebit. Propositionem igitur mihi fuerat locum hujus poli accuratus investigare, pro eoque quantitatem coordinatarum $AC$ et $CO$ perscrutari.

§. 126. Hunc in finem, introducta in calculum portionis cujusvis $AM = s$ amplitudine $= \Phi$, ut sit $r = \frac{ds}{d\Phi}$, sit $sds = aad\Phi$, hincque

$$ss = 2a\Phi,$$

et $s = a\sqrt{2\Phi} = 2c\sqrt{\Phi}$. Hinc jam profit $\frac{ds}{d\Phi} = \frac{c^2\Phi}{\sqrt{\Phi}}$, unde posita absissa pro hoc arcu $AP = x$ et applicata $PM = y$, colligitur fore

$$x = c\int \frac{d\Phi}{\sqrt{\Phi}},$$

$$y = c\int \frac{\sin\Phi}{\sqrt{\Phi}}.$$

§. 127. Hine ergo pro polo $O$ determinando requi- runtur valores harum duarum formularum integralium, post- quam a termino $\Phi = \theta$ usque ad $\Phi = \infty$ fuerint extensae. Inuito quidem sum arbitrus, hos valores aliter obtineri non
AD TOM. I. CAP. VIII.

posse nisi approximando, dum utraque formula successive per partes evolvatur; primo scilicet a $\Phi = 0$ usque ad $\Phi = \pi$; inde a $\Phi = \pi$ usque ad $\Phi = 2\pi$; porro a $\Phi = 2\pi$ usque ad $\Phi = 3\pi$; etc. quippe quo pacto series prodibunt satis prompte convergentes. Verum evidens est hanc operationem longos calculos satis taediosos requirere, quos quidem evolvere non sum ausus. Nuper autem forte fortuna per methodum prorsus singularum perspexi esse tam

$$\int \frac{\partial \cos \Phi}{\sqrt{\Phi}} \left[ \frac{a}{ad} \Phi \to 0 \right] = \sqrt{\frac{\pi}{2}} \text{ quam}$$

$$\int \frac{\partial \sin \Phi}{\sqrt{\Phi}} \left[ \frac{a}{ad} \Phi \to 0 \right] = \sqrt{\frac{\pi}{2}}$$

ita ut pro loco poli quaesito $O$ sit

$$AC = c \sqrt{\frac{\pi}{2}} \text{ et } CO = c \sqrt{\frac{\pi}{2}}$$

§ 128. Quoniam igitur methodus, qua huc sum perductus, non parum pollacri videtur, Geometris haud ingratum fore arbitror, si eam omni cura hic exponuero. Et quia multo latius quam ad istas formulas patet, eam etiam omnia extensione sum propositurus, quae omnia ex consideratione hujus formulae satis simplicis $\int x^{n-1} \partial x e^{-x}$ deduxi, cujus ergo integrale pro variis valoribus exponentis $n$ investigare convenit.

§ 129. Ac primo quidem, pro casu $n = 1$ hujus formulae $\int \partial x e^{-x}$ integrale manifestum est $1 - e^{-x}$, quod casu $x = 0$ evanescit, facto autem $x = \infty$ abit in unitatem. Praeterea, cum hujus formulae $x^\lambda \cdot e^{-x}$ differentiale sit

$$\lambda x^{\lambda-1} \partial x \cdot e^{-x} = x^{\lambda} \partial x \cdot e^{-x}$$

erit vicissim

$$\int x^\lambda \partial x \cdot e^{-x} = \lambda \int x^{\lambda-1} \partial x \cdot e^{-x} = x^{\lambda} \cdot e^{-x}$$

quod postremum membrum tam pro casu $x = 0$ quam $x = \infty$ evanescit, si modo fuerit $\lambda > 0$. Tam igitur pro nostris ter-
340. SUPPLEMENTUM V.

minis integrationis erit
\[ \int x^\lambda \, \partial x \cdot e^{-x} = \lambda \int x^{\lambda-1} \, \partial x \cdot e^{-x}, \]
cujus formulae opus, ob \( \int \partial x \, e^{-x} = 1 \), sequentes integralium valores deducuntur
\[ \int x \, \partial x \cdot e^{-x} = 1 \]
\[ \int x^2 \, \partial x \cdot e^{-x} = 1 \cdot 2 \]
\[ \int x^3 \, \partial x \cdot e^{-x} = 1 \cdot 3 \cdot 3 \]
\[ \int x^4 \, \partial x \cdot e^{-x} = 1 \cdot 2 \cdot 3 \cdot 4 \]
sicque in genere
\[ \int x^{n-1} \, \partial x \cdot e^{-x} = 1 \cdot 2 \cdot 3 \cdot 4 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (n-1), \]
cujus producti valores quoties \( n \) fuerit numerus integer positivus sponte se produunt; quando autem \( n \) est numerus fractus olim ostendi, quomodo valores per quadraturas curvarum algebraicarum exhiberi queant. Sic pro caso \( n = \frac{1}{2} \) constat, istum valorem esse \( \sqrt{\pi} \).

§ 130. Cum igitur omnes valores hujus producti infiniti 1 \cdot 2 \cdot 3 \cdot 4 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (n-1) tanquam cogniti spectari debeant, eos littera \( \Delta \) designabo, ita ut sit \( \Delta = 1 \cdot 2 \cdot 3 \cdot 4 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (n-1) \), sicque jam adepti sumus hanc insignem formulam integralem
\[ \int x^{n-1} \, \partial x \cdot e^{-x} = \Delta, \]
integrali scilicet ab \( x = 0 \) ad \( x = \infty \) extenso; atque ex hac ipsa formula omnia deduxi, quae ad casum ante memoratum pertinent, ubi quidem ratiocinia penitus singularia adhiberi debent, quae igitur hic diligentius sum expositurus.

§ 131. Posui autem primo \( x = ky \), et quoniam ambo termini integralis iisdem manerit, erit etiam
\[ k^n \int y^{n-1} \, \partial y \cdot e^{-ky} = \Delta, \]
quandoquidem haec formula etiam ab \( y = 0 \) ad \( y = \infty \) usque
extenditur; quamobrem per $k^n$ dividendo habebimus

$$f y^{n-1} \partial y \cdot e^{-k y} = \frac{\Delta}{k^n},$$

ubi autem notari oportet, pro $k$ nullos numeros negativos accipi posse, quia aliquo quin formula $e^{-k y}$ non amplius evanescet casu $x = 0$, atque his isti soli valores sunt excludendi, ita ut etiam valores imaginarii loco $k$ adhiberi queant, atque hinc illas arduas integrationes sum assecatus.

§. 132. Ponamus ergo $k = p + q \sqrt{-1}$, et cum sit

$$e^{-p y - q y} = \cos q y - \sqrt{-1} \sin q y,$$

$$e^{+p y + q y} = \cos q y + \sqrt{-1} \sin q y,$$

nstra formula nunc induet hanc formam

$$f y^{n-1} \partial y \cdot e^{-p y} \left( \cos q y - \sqrt{-1} \sin q y \right) = \frac{\Delta}{(p + q \sqrt{-1})^n}.$$

Quamobrem si formulae imaginariae signum mutemus, erit simili modo

$$f y^{n-1} \partial y \cdot e^{-p y} \left( \cos q y + \sqrt{-1} \sin q y \right) = \frac{\Delta}{(p - q \sqrt{-1})^n}.$$

§. 133. Quo valores inventos commodius exprimere licet, ponamus $p = f \cos \theta$ et $q = f \sin \theta$, eritque

$$(p + q \sqrt{-1})^n = f^n \left( \cos n \theta + \sqrt{-1} \sin n \theta \right)$$

et

$$(p - q \sqrt{-1})^n = f^n \left( \cos n \theta - \sqrt{-1} \sin n \theta \right);$$

ubi notasse juvabit fore tang. $\theta = \frac{q}{p}$, unde ex valoribus $p$ et $q$ assumptis erit etiam $f = \sqrt{(pp + qq)}$. Hoc ergo modo sit priore casu

$$\frac{\Delta}{(p + q \sqrt{-1})^n} = f^n \left( \cos n \theta + \sqrt{-1} \sin n \theta \right);$$
SUPPLEMENTUM V.

pro altero
\[
\frac{\Delta}{(p-q\sqrt{-1})^2} = \frac{\Delta}{f^n (\cos n\theta - \sqrt{-1} \sin n\theta)}
\]
Quamobrem si hae duae formulae addantur probibit
\[
2 \frac{\Delta \cos n\theta}{f^n} = \frac{\Delta}{f^n}.
\]
Differentia autem harum formularum dat
\[
2 \frac{\sqrt{-1} \sin n\theta}{f^n}.
\]

§. 134. Addamus igitur quoque ipsas formulas integrales, et habebimus
\[
\int y^{n-1} \partial y \cdot e^{-pq} \cos qy = \frac{\Delta \cos n\theta}{f^n}.
\]
Sin autem subtrahamus et per \(2\sqrt{-1}\) dividamus, oritur
\[
\int y^{n-1} \partial y \cdot e^{-pq} \sin qy = \frac{\Delta \sin n\theta}{f^n}.
\]
Quae jam duae formulae integrales latissime patent, cum numeri \(p\) et \(q\) prorsus arbitrio nostro relinquentur, id tantum observando, ne pro \(p\) numeri negativi accipiantur. Operae igitur pretium erit, has duas formulas integrales sequentibus binis theorématis complecti.

Theorema I.

Posito \(\Delta = 1 \cdot 2 \cdot 2 \cdot \ldots \cdot (n-1)\), et pro litteris \(p\) et \(q\) numeros quoscunque positivos accipiendo, fiat inde \(\sqrt{(pp + qq)} \equiv f\); et quaeatur angulus \(\theta\), ut sint tang. \(\theta = \frac{q}{p}\); et habebitur ista integratio memorabilis
\[
\int x^{n-1} \partial x \cdot e^{-px} \cos qx \begin{bmatrix} \text{ab} & x \equiv 0 \end{bmatrix} = \frac{\Delta \cos n\theta}{f^n}.
\]
Theoremata II.

Posito $\Delta = 1.2.3\ldots (n-1)$, et pro literis $p$ et $q$ numeros quoscunque positivos accipiendo, fiat inde $\sqrt{(pp + qq)} = f$, et quaeratur angulus $\theta$, ut sit tang. $\theta = \frac{q}{p}$, atque habebitur ista integratio memorabilis

$$\int_{x=0}^{x=\infty} \frac{x^{n-1} \partial x \cdot e^{-px} \sin qx}{f^n} = \frac{\Delta \sin n\theta}{f^n}.$$

§. 135. Cum igitur pro casu curvae supra consideratae pervenerimus ad has formulas integrales

$$\frac{\int \frac{\partial n}{\sqrt{\partial n}}}{\sqrt{\partial n}} \text{ et } \frac{\int \frac{\partial n}{\sqrt{\partial n}}}{\sqrt{\partial n}},$$

facta applicatione erit $n = \frac{1}{2}$, ideoque $\Delta = \sqrt{\pi}$, tum vero erit $p = 0$ et $q = 1$, unde fit $f = 1$ et tang. $\theta = \frac{q}{p} = \infty$; ideoque $\theta = \frac{\pi}{2}$, ergo $\cos n\theta = \frac{1}{\sqrt{2}} = \sin n\theta$. Hinc igitur fit

$$\int \frac{\partial n}{\sqrt{\partial n}} \left[ \frac{a \phi = 0}{\phi = \infty} \right] = \sqrt{\frac{\pi}{2}}, \text{ simulque}$$
$$\int \frac{\partial n}{\sqrt{\partial n}} \left[ \frac{a \phi = 0}{\phi = \infty} \right] = \sqrt{\frac{\pi}{2}}.$$

§. 136. Operae autem pretium erit, hunc casum quo $n = \frac{1}{2}$ et $\Delta = \sqrt{\pi}$ in genere evolvere, et eum posuerimus

$$\sqrt{(pp + qq)} = f$$
$$\text{et } \frac{q}{p} = \text{ tang. } \theta,$$
$$\text{erit}$$
$$\sin \theta = \frac{q}{f} \text{ et } \cos \theta = \frac{p}{f}. $$

Hinc ergo primo

$$\sin \frac{1}{2} \theta = \sqrt{\frac{f - \cos \theta}{2}} = \sqrt{\frac{f - p}{2f}} \text{ et}$$
$$\cos \frac{1}{2} \theta = \sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{f + p}{2f}};$$

unde fit pro valoxibus integralibus

$$\frac{\Delta \sin \frac{1}{2} \theta}{\sqrt{f}} = \sqrt{\frac{\pi}{f}} \sqrt{\frac{f - p}{2}} \text{ et}$$
\[ \frac{\Delta \cos \frac{1}{2} \theta}{\sqrt{f}} = \frac{\sqrt{n}}{f} \cdot \frac{\sqrt{f-p}}{2}. \]

Quamobrem habebimus binas sequentes formulæ integrales
\[ \int \frac{\partial x}{\sqrt{f}} e^{-px} \sin qx \, dx = \frac{\sqrt{n}}{f} \cdot \frac{\sqrt{f-p}}{2}, \]
\[ \int \frac{\partial x}{\sqrt{f}} e^{-px} \cos qx \, dx = \frac{\sqrt{n}}{f} \cdot \frac{\sqrt{f+p}}{2}. \]

§. 137. Casus autem, quibus pro \( n \) sumitur numerus integer positivus, ideoque \( \Delta \) absolute per numeros integros exhiberì potest, ita sunt comparati, ut etiam per methodos cognitias, ope scilicet formularum integralium reductionis satís notae expediri queant, atque adeo integralia in genere exhiberi. Haec autem operatio postulat calculos non parum prolixos, quamobrem formæ nostrae satis simplices pro casu scilicet \( x = \infty \) nihilominus omni atttentione sunt dignae. Quando autem exponenti \( n \) valores negativos tribuere voluerimus, hi casus statim in initio integrationis additionem constantis infinitae postulant, ut scilicet integralia evanescant casu \( x = 0 \), sique adeo valores integralium, quae hic quaeque ad institutum nostrum non sunt referendi.

§. 138. Casus autem maxime memorabilis hic occurrit, quo \( n = 0 \), et qui prorsus singularem sollertiam postulat, quem igitur accuratus evolvamus. Quodiam posuimus
\[ \Delta = 1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1), \]
statuamus simili modo
\[ \Delta' = 1 \cdot 2 \cdot 3 \ldots n, \text{ et } \Delta'' = 1 \cdot 2 \cdot 3 \ldots \ldots (n+1), \]
crevitque manifesto
\[ \Delta = \frac{\Delta'}{n}, \text{ et } \Delta' = \frac{\Delta''}{n+1}, \text{ ideoque } \Delta = \frac{\Delta''}{n(n+1)}. \]
Sumamus nunc \( n = \omega \), existente \( \omega \) infinite parvo, et cum sit
AD TOM. I. CAP. VIII.

Δ'' = 1, unde fit Δ = \frac{\pi}{2}, ideoque ejus valor erit infinitus. Cum autem pro formula integrali prior si sit sin. n₀ = \omega \theta, evidens est fore Δ sin. n₀ = \theta; quamobrem ista prior formula integralis erit \int \frac{2x}{\pi} e^{-px} \sin. q x = \theta, dum nempe integrale a termino x = 0 usque ad terminum x = \infty extenditur. Alterius autem formae nostrae integralis \int \frac{2x}{\pi} e^{-px} \cos. q x valor erit infinite magnus. Ille autem casus omnino meretur ut eum singulari theoremate complectamur.

Theoremata III.

§ 139. Si litterae p et q denotent numeros positivos quoscunque, atque hinc quae rum angulus \theta, ut sit tang. \theta = \frac{q}{p}, habebit sequens integratio maxime memorabilis

\int \frac{2x}{\pi} e^{-px} \sin. q x \left[ \begin{array}{cc} ab x = 0 \\ ad x = \infty \end{array} \right] = \theta

cujus theorematis demonstratio dubito quin alio modo quam per approximationes investigari queat.

§ 140. Casus autem simplicissimus quo p = 0 et q = 1 jam omnia calculi artificio adhuc cognita superare videtur, quia autem hoc casu sit tang. \theta = \frac{1}{2} = \infty, erit \theta = \frac{\pi}{2}, unde oritur haec integratio \int \frac{2x}{\pi} \sin. x = \frac{\pi}{2}. Interim tamen de ejus veritate eo minus dubitare licet, quod approximationes adhibitaæ ad eundem valorem propemodum perducant. Quodsi hunc casum eum initio memorato \int \frac{2x}{\pi} \sin. x = \sqrt{\frac{\pi}{2}} comparemus, ingens similitudo summam attemdem meretur, cum hujus integrale sit praecise radix quadrata illius.
5) **Investigatio formulae integralis** }\( \int \frac{x^{m-1}}{(1+x^k)^n} \, dx \), **casu quo post integrationem statuitur** }\( x = \infty \). **Opuscula Analytica.** **Tom. II. Pag. 42 — 54.**

§ 141. Jam satis notum est, hujus formulae integrale-partim logarithmos, partim arcus circulares complecti, et partes logarithmicas hanc progressiones constituere

\[
- \frac{2}{k} \cos \frac{m \pi}{k} \, l \sqrt{1 - 2x \cos \frac{\pi}{k} + xx} \\
- \frac{2}{k} \cos \frac{3m \pi}{k} \, l \sqrt{1 - 2x \cos \frac{3\pi}{k} + xx} \\
- \frac{2}{k} \cos \frac{5m \pi}{k} \, l \sqrt{1 - 2x \cos \frac{5\pi}{k} + xx} \\
- \frac{2}{k} \cos \frac{7m \pi}{k} \, l \sqrt{1 - 2x \cos \frac{7\pi}{k} + xx} \\
- \ldots \\
- \frac{2}{k} \cos \frac{im \pi}{k} \, l \sqrt{1 - 2x \cos \frac{i \pi}{k} + xx}
\]

ubi }\( i \) denotat numerum imparem non majorem quam }\( k \). Hinc si }\( k \) fuerit numerus par, erit }\( i = k - 1 \); ac si }\( k \) fuerit numerus impar, hanc progressiones continuari oportet usque ad }\( i = k \), ejus vero coefficientes duplo minor capi debet, seu loco }\(-\frac{2}{k}\) tantum scribi debet }\(-\frac{1}{k}\), cujus irregularitatis ratio in Tomo I est exposita.

§ 142. Cum hae partes sponte jam evanescant in posito }\( x = 0 \), statuamus statim }\( x = \infty \), et cum in genere sit

\[
\sqrt{1 - 2x \cos \omega + xx} = x - \cos \omega, \text{ erit} \\
\sqrt{1 - 2x \cos \omega + xx} = l(x - \cos \omega) \\
= lx - \frac{\cos \omega}{x} = lx, \text{ ob } \frac{\cos \omega}{x} = 0;
\]

omnes ergo illi logarithmi reducuntur ad eandem formam }\( lx \),
AD TOM. I. CAP. VIII.

quae multiplicanda est per hanc seriem

\[-\frac{2}{k} \cos \frac{m\pi}{k} + \frac{2}{k} \cos \frac{3m\pi}{k} - \frac{2}{k} \cos \frac{5m\pi}{k} + \ldots - \frac{2}{k} \cos \frac{im\pi}{k},\]

ubi, ut diximus, \(i\) denotat maximum numerum imparem ipso \(k\) non majorem, hac tamen restrictione, ut, si \(k\) fuerit impar, ideoque \(i = k\), ultimum membro ad dimidium reduci debet. Quamobrem, si hujus progressionis summam investigare velimus, duo casus erunt constitutendi: alter quo \(k\) est numerus par et \(i = k - 1\), alter vero quo \(k\) est impar et \(i = k\).

Evolutio casus prioris, quo \(k\) est numerus par et \(i = k - 1\).

§ 143. Hoc ergo casu, posito \(x = \infty\), formula \(-\frac{2}{k} \frac{\sin m\pi}{k}\) multiplicatur per hanc eosinum seriem

\[\cos \frac{m\pi}{k} + \cos \frac{3m\pi}{k} + \cos \frac{5m\pi}{k} + \ldots + \cos \frac{(k-1)m\pi}{k};\]

cujus summam statuamus \(= S\). Ducamus hanc seriem in sin. \(\frac{m\pi}{k}\), et cum in genere sit

\[\sin \frac{m\pi}{k} \cos \frac{i\pi}{k} = \frac{1}{2} \sin \frac{(i+1)m\pi}{k} - \frac{1}{2} \sin \frac{(i-1)m\pi}{k},\]

facta hae reductione habebimus

\[S \sin \frac{m\pi}{k} = \frac{1}{2} \sin \frac{2m\pi}{k} + \frac{1}{2} \sin \frac{4m\pi}{k} + \ldots + \frac{1}{2} \sin \frac{(k-1)m\pi}{k} + \frac{1}{2} \sin m\pi\]

\[-\frac{1}{2} \sin \frac{2m\pi}{k} - \frac{1}{2} \sin \frac{4m\pi}{k} - \ldots - \frac{1}{2} \sin \frac{(k-2)m\pi}{k} - \frac{1}{2} \sin \frac{(k-1)m\pi}{k};\]

ubi omnes termini praeter ultimum manifesto se destruunt, ita ut sit

\[S \sin \frac{m\pi}{k} = \frac{1}{2} \sin m\pi.\]

Jam vero quia nostri coefficientes \(m\) et \(k\) supponuntur integri, utique est sin. \(m\pi = 0\), ideoque etiam \(S = 0\), nisi forte etiam fuerit sin. \(\frac{m\pi}{k} = 0\), qui autem casus locum habere nequit, quoniam in integratione formulae propositae

\[\frac{x^{m-1}}{1 + x^k}\]

44.
semper assumi solet esse \( n < k \). Hoc igitur modo evictum est, casu quo post integrationem \( x = \infty \), omnes partes logarithmicas integralis se dæstruere.

Evolutio casus alterius, quo est \( k \) numerus impar

et \( i = k \).

§ 144. Hoc ergo casu, sumto \( x = \infty \), formula \( l \) multiplicatur per hanc seriem

\[
-\frac{2}{k} \cos \frac{m\pi}{k} - \frac{2}{k} \cos \frac{3m\pi}{k} - \frac{2}{k} \cos \frac{5m\pi}{k} \cdots - \frac{2}{k} \cos \frac{\pi}{k},
\]

ubi terminus penultimus est \( -\frac{2}{k} \cos \frac{(k-2)m\pi}{k} \), pro ultimo vero termino erit \( \cos \frac{\pi}{k} = 1 \), signo superiore valente si \( m \) sit numerus par; inferiori si impar; quare remoto termino ultimo pro reliquis ponamus

\[
\cos \frac{\pi}{k} + \cos \frac{3\pi}{k} + \cos \frac{5\pi}{k} + \cdots + \cos \frac{(k-2)\pi}{k} = S,
\]

ita ut multiplicator ipsius logarithmi \( x \) sit

\[
-\frac{2S}{k} = \frac{\pi}{k} \cos \frac{\pi}{k}.
\]

Hinc procedendo ut ante fiet

\[
S \sin \frac{\pi}{k} = \frac{1}{2} \sin \frac{2\pi}{k} + \frac{1}{2} \sin \frac{4\pi}{k} + \frac{1}{2} \sin \frac{6\pi}{k} + \cdots + \frac{1}{2} \sin \frac{(k-2)\pi}{k} + \frac{1}{2} \sin \frac{(k-1)\pi}{k}.
\]

ubi iterum omnes termini praeter ultimum se mutuo tollunt, ita ut hinc procedat

\[
S \sin \frac{\pi}{k} = \frac{1}{2} \sin \frac{(k-1)\pi}{k} = \frac{1}{2} \sin \left( \frac{\pi}{k} - \frac{\pi}{k} \right);
\]

at vero est

\[
\sin \left( \frac{\pi}{k} - \frac{\pi}{k} \right) = \sin \frac{\pi}{k} \cos \frac{\pi}{k} - \cos \frac{\pi}{k} \sin \frac{\pi}{k},
\]

ubi notetur esse \( \sin \frac{\pi}{k} = 0 \), ob \( m \) numerum integrum; habebimus ergo

\[
S \sin \frac{\pi}{k} = -\frac{1}{2} \cos \frac{\pi}{k} \sin \frac{\pi}{k}, \text{ sive } S = -\frac{1}{2} \cos \frac{\pi}{k},
\]
sequentem multiplicat½ ipsius \( x \) erit
\[
\frac{1}{k} \cos \frac{m \pi}{k} - \frac{1}{k} \cos \frac{m \pi}{k} = 0,
\]
sicque manifestum est, sive \( k \) sit numerus par sive impar, omnia
membra logarithmica in nostro integrali se mutuo destrueri, si-
quidem post integrationem statuamus \( x = \infty \), quemadmodum hic
semper supponimus.

§ 145. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulæ componitur. Hae autem partes sequentem progressionem constituere sunt compertae

\[
\begin{align*}
\frac{2}{k} \sin \frac{m \pi}{k} \text{ Arc. tang.} \frac{x \sin \frac{\pi}{k}}{1 - x \cos \frac{\pi}{k}} & + \frac{2}{k} \sin \frac{3m \pi}{k} \text{ Arc. tang.} \frac{x \sin \frac{3\pi}{k}}{1 - x \cos \frac{3\pi}{k}} \\
\frac{2}{k} \sin \frac{5m \pi}{k} \text{ Arc. tang.} \frac{x \sin \frac{5\pi}{k}}{1 - x \cos \frac{5\pi}{k}} & + \frac{2}{k} \sin \frac{7m \pi}{k} \text{ Arc. tang.} \frac{x \sin \frac{7\pi}{k}}{1 - x \cos \frac{7\pi}{k}} \\
\frac{2}{k} \sin \frac{9m \pi}{k} \text{ Arc. tang.} \frac{x \sin \frac{9\pi}{k}}{1 - x \cos \frac{9\pi}{k}} & + \frac{2}{k} \sin \frac{11m \pi}{k} \text{ Arc. tang.} \frac{x \sin \frac{11\pi}{k}}{1 - x \cos \frac{11\pi}{k}}
\end{align*}
\]

ubi in ultimo membro est vel \( i = k - 1 \), vel \( i = k \); prius scilicet valet si \( i \) est numerus par, posterius si impar.

§ 146. Cum etiam omnia haece membra evanescant
posito \( x = 0 \), faciamus pro instituto nostro \( x = \infty \). In genere
igitur fit

\[
\text{Arc. tang.} \frac{x \sin \frac{i \pi}{k}}{1 - x \cos \frac{i \pi}{k}} = \text{Arc. tang.} \left( -\text{tang.} \frac{i \pi}{k} \right).
\]

Est vero

\[
-\text{tang.} \frac{i \pi}{k} = -\text{tang.} \frac{(k - i) \pi}{k},
\]

ex quo hic arcus fit \( \frac{(k - i) \pi}{k} \). Hinc ergo loco \( i \) scribend\ö
successive numeros \(1, 3, 5, 7\) etc. istae partes nostri integra-
is quae sit erunt
\[
\frac{2}{k} \left(\frac{k-1}{k}\right) \pi \sin \frac{2m\pi}{k} + \frac{2}{k} \left(\frac{k-3}{k}\right) \pi \sin \frac{3m\pi}{k} + \frac{2}{k} \left(\frac{k-5}{k}\right) \pi \sin \frac{5m\pi}{k} + \frac{2}{k} \left(\frac{k-7}{k}\right) \pi \sin \frac{7m\pi}{k} + \frac{2}{k} \left(\frac{k-9}{k}\right) \pi \sin \frac{9m\pi}{k} + \ldots + \frac{2}{k} \left(\frac{k-i}{k}\right) \pi \sin \frac{im\pi}{k}
\]
ubi casu, quo \(k\) est numerus par, progressi oportet usque ad
\(i = k - 1\); ac si \(k\) sit numerus impar, usque ad \(i = k\).

§. 147. Statuamus brevitates gratia
\[
\left(\frac{k-1}{k}\right) \pi \sin \frac{m\pi}{k} + \left(\frac{k-3}{k}\right) \pi \sin \frac{3m\pi}{k} + \left(\frac{k-5}{k}\right) \pi \sin \frac{5m\pi}{k} + \ldots + \left(\frac{k-i}{k}\right) \pi \sin \frac{im\pi}{k} = S
\]
ita ut integrale quae sit \(\frac{2\pi}{kk''}\) quandoquidem partes logarih-
micae se mutuo destruxerunt. Multiplicemus nunc utrique per
\(2 \sin \frac{m\pi}{k}\), et cum in genere sit
\[
2 \sin \frac{m\pi}{k} \sin \frac{im\pi}{k} = \cos \frac{(i-1)m\pi}{k} - \cos \frac{(i+1)m\pi}{k},
\]
facta substitutione erit
\[
2S \sin \frac{m\pi}{k} = \left(\frac{k-1}{k}\right) \cos \frac{0m\pi}{k} + \left(\frac{k-3}{k}\right) \cos \frac{2m\pi}{k} + \left(\frac{k-5}{k}\right) \cos \frac{4m\pi}{k} + \ldots + \left(\frac{k-i}{k}\right) \cos \frac{(i-1)m\pi}{k} - \left(\frac{k-i}{k}\right) \cos \frac{(i+1)m\pi}{k}
\]
quae series manifesto contrahitur in sequentem
\[
2S \sin \frac{m\pi}{k} = \left(\frac{k-1}{k}\right) - 2 \cos \frac{2m\pi}{k} + 2 \cos \frac{4m\pi}{k} - 2 \cos \frac{6m\pi}{k} + \ldots + \left(\frac{k-i}{k}\right) \cos \frac{(i-1)m\pi}{k} - \left(\frac{k-i}{k}\right) \cos \frac{(i+1)m\pi}{k}
\]
ubi, primo et ultimo membro sublatis, regularem termini inter-
medii constitutum seriem, pro cuius valore investigando ponamus
\[
T = \cos \frac{2m\pi}{k} + \cos \frac{4m\pi}{k} + \cos \frac{6m\pi}{k} + \ldots + \cos \frac{(i-1)m\pi}{k},
\]
ita ut sit
\[ 2S \sin \frac{m\pi}{k} = k - 4 - 2T - (k - 4) \cos \frac{(i-1)m\pi}{k}. \]

Hic autem iterum convenit duos casus perpendere, prout \( k \) facit par vel impar.

Evolutio casus prioris, quo \( k \) est numerus par et \( i = k - 4 \).

§ 148. Hec ergo casu habebimus
\[ T = \cos \frac{2m\pi}{k} + \cos \frac{4m\pi}{k} + \cos \frac{6m\pi}{k} + \ldots + \cos \frac{(k-2)m\pi}{k}. \]

Multiplicemus denuo per \( 2 \sin \frac{m\pi}{k} \), et per reductiones supra indicatas habebimus
\[ 2T \sin \frac{m\pi}{k} = \sin \frac{3m\pi}{k} + \sin \frac{5m\pi}{k} + \ldots + \sin \frac{(k-3)m\pi}{k} + \sin \frac{(k-1)m\pi}{k}. \]

Deletis igitur terminis se mutuo tollentibus erit
\[ 2T \sin \frac{m\pi}{k} = - \sin \frac{m\pi}{k} - \sin \frac{(k-1)m\pi}{k}. \]

Est vero
\[ \sin \frac{(k-1)m\pi}{k} = \sin \left( m\pi - \frac{m\pi}{k} \right) = \sin m\pi \cos \frac{m\pi}{k} - \cos m\pi \sin \frac{m\pi}{k}, \]
ubi \( \sin m\pi = 0 \), quamobrem fit \( 2T = 1 - \cos m\pi \).

§ 149. Invento valore pro \( T \) colligitur fore
\[ 2S \sin \frac{m\pi}{k} = k, \]ideoque \( S = \frac{k}{2 \sin \frac{m\pi}{k}}. \)

Denique vero ipse valor formulæ nostræ integralis, quem quae- rimus, erit \( \frac{2\pi S}{kk} \), et nunc manifestum est, integrale nostræ formu- lae, casu quo \( S \) est numerus par, fore \( \frac{\pi}{k \sin \frac{m\pi}{k}} \), siquidem post integrationem statuatur \( x = \infty \).
SUPPLEMENTUM V.

Evolutio alterius casus, quo $k$ est numerus impar et $i = k$.

§ 150. Hoc ergo casu est

$$T = \cos \frac{2m\pi}{k} + \cos \frac{4m\pi}{k} + \cos \frac{6m\pi}{k} + \ldots + \cos \frac{(k-1)m\pi}{k},$$

quae series multiplicata per $2\sin \frac{m\pi}{k}$ producet ut ante

$$2T \sin \frac{m\pi}{k} = \ldots - \sin \frac{3m\pi}{k} + \sin \frac{5m\pi}{k} - \ldots - \sin \frac{(k-2)m\pi}{k} + \sin \frac{km\pi}{k}$$

unde deletis terminis se mutuo tollentibus reperietur

$$2T \sin \frac{m\pi}{k} = \sin \frac{m\pi}{k} + \sin m\pi$$

ideoque

$$2T = 1 + \frac{\sin m\pi}{\sin \frac{m\pi}{k}} = 1,$$

ob $\sin m\pi = 0$,

hincque porro set:

$$2S \sin \frac{m\pi}{k} = k;$$

quare cum valor integralis quae sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $\frac{\pi}{k \sin \frac{m\pi}{k}}$, prorsus uti praecedente casu.

Hinc ergo deducimus sequens theorema.

Theorema.

§ 151. Si haec formula differentialis $\frac{\pi}{k \sin \frac{m\pi}{k}}$ ita integratur, ut, posito $x = 0$, integrale evanesceat, tum vero statuetur $x = \infty$, valor inde resultans semper erit $\frac{\pi}{k \sin \frac{m\pi}{k}}$, sive $k$ sit numerus par, sive impar. Hujus theoremati demonstratio ex praecedentibus est manifesta.
§ 152. In evolutione hujus formulæ assumsimus esse \( m < k \), quia alioquin membra logarithmica se non destruissent; at vero ne haec quidem limitatione nunc amplius est opus. Casu enim quo foret \( m = k \), integrale formulæ \( \frac{x^{m-1} \partial x}{1 + x^k} \) esset \( \frac{k}{k} \int \left(1 + x^k\right) \), quod facit \( x = \infty \) fieret etiam \( \infty \); verum hoc idem indicat, nostrum integrale esse \( \frac{\pi}{k \sin \pi} = \infty \). Dummodo ergo \( m \) non fuerit majus quam \( k \), nostra formula veritati semper est consentanea.

§ 153. Quin etiam ne quidem nécesse est ut exponentes \( m \) et \( k \) sint numeri integri, dummodo non fuerit \( m > k \); si enim fuerit \( m = \frac{k}{\lambda} \) et \( k = \frac{x}{\lambda} \), erit valor per nostram formulam \( \frac{\lambda \tau}{\kappa \sin \frac{\mu \tau}{\kappa}} \), cuius veritas ita ostenditur. Quia hoc casu formula integranda est \( \int \frac{y^\mu}{1 + y^k} \cdot \frac{\partial y}{y} \), statuatur \( x = y^\lambda \), erit \( \frac{\partial x}{x} = \frac{\lambda \partial y}{y} \), et formula fiet

\[ \int \frac{y^\mu}{1 + y^k} \cdot \frac{\lambda \partial y}{y} = \lambda \int \frac{y^{\mu - 1}}{1 + y^k} \cdot \frac{\partial y}{y} \]

cuius valor utique erit \( \frac{\lambda \tau}{\kappa \sin \frac{\mu \tau}{\kappa}} \).

Alia demonstrationi theorematis.

§ 154. Denotet \( P \) valorem integralis \( \int \frac{x^m}{1 + x^k} \cdot \frac{\partial x}{x} \) a termino \( x = 0 \) usque ad \( x = 1 \); at \( Q \) valorem ejusdem integralis a termino \( x = 1 \) usque ad \( x = \infty \), ita ut \( P + Q \) praebat eum ipsum valorem; qui in theoremate continentur.

Vol. IV.
SUPPLEMENTUM V.

Nunc pro valore $Q$ inveniendo statuat $x = \frac{1}{y}$, unde fit
\[ \frac{\partial x}{x} = -\frac{\partial y}{y}, \]
sectue
\[ Q = \int \frac{y^{m}}{1+y^{k}} \cdot \frac{-\partial y}{y} = -\int \frac{y^{k-m}}{1+y^{k}} \cdot \frac{\partial y}{y} \]
a termino $y = 1$ usque ad $y = 0$. Hinc igitur commutatis
terminis erit
\[ Q = +\int \frac{y^{k-m}}{1+y^{k}} \cdot \frac{\partial y}{y} \]
a termino $y = 0$ usque ad $y = 1$. Jam quia hoc integrálí
expedito littera $y$ ex calculo egreditur, loco $y$ scribere licebit $x$,
ita ut sit
\[ Q = \int \frac{x^{k-m}}{1-x^{k}} \cdot \frac{\partial x}{x}, \]
quo facto habebimus
\[ P + Q = \int \frac{x^{m} + x^{k-m}}{1+x^{k}} \cdot \frac{\partial x}{x} \]
a termino $x = 0$ usque ad terminum $x = 1$. Verum non
ita pridem demonstravi, valorem hujus formulae integrális intra
termínos $x = 0$ et $x = 1$ contentum esse $= \frac{\pi}{k \sin \frac{m\pi}{k}}$.

Hinc igitur nascitur sequens theórema non minus notatu dignum.

Theórem a.

§ 165. Valore hujus formulae integrális
\[ \int \frac{x^{m} + x^{k-m}}{1+x^{k}} \cdot \frac{\partial x}{x} \]
intra terminíos $x = 0$ et $x = 1$ contentus, aequalis est valorí
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istius integralis \( \int \frac{x^{m-1}}{1+x^k} \cdot \frac{\partial x}{x} \), intra terminos \( x = 0 \) et \( x = \infty \) contento.

§. 156. Hæc expensis formulam integralem in titulo propositione aggregiamur, et quo eam ad formam haec tunam tractatam reducamus, in subsidium vocem sequentem reductionem

\[
\int \frac{x^{m-1}}{(1+x^k)^{\lambda+k}} \partial x = \frac{Ax^m}{(1+x^k)^{\lambda}} + B \int \frac{x^{m-1}}{(1+x^k)^{\lambda}} \partial x,
\]

unde facta differentiatione prodit sequens aequatio

\[
\frac{x^{m-1}}{(1+x^k)^{\lambda+1}} = \frac{mAx^{m-1}}{(1+x^k)^{\lambda}} - \frac{\lambda k A x^{m+k-1}}{(1+x^k)^{\lambda+1}} + \frac{Bx^{m-1}}{(1+x^k)^{\lambda}}.
\]

quae aequatio per \( x^{m-1} \partial x \) divisa ac per \( (1+x^k)^{\lambda} \) multipli
cata, terminum negativum a dextra ad sinistram transponendo, erit

\[
1 + \lambda k A x^k = mA + B,
\]

quae aequatio manifesto subsistere nequit, nisi sit \( \lambda k A = 1 \),
sive \( A = \frac{1}{\lambda k} \), unde erit \( 1 = mA + B = \frac{m}{\lambda k} + B \), sicque

\( B = 1 - \frac{m}{\lambda k} \).

§. 157. Inventis his valoribus pro litteris \( A \) et \( B \), primum assumimus, integralia ita capi, ut evanescant posito \( x = 0 \);
tum vero posito \( x = \infty \), quia exponens \( n \) minor supponitur quam \( k \), membro absolutione littera \( A \) affectum sponte evanesceit, ita ut hoc casu \( x = \infty \) fiat

\[
\int \frac{x^{m-1}}{(1+x^k)^{\lambda+1}} \partial x = \left(1 - \frac{m}{\lambda k}\right) \int \frac{x^{m-1}}{(1+x^k)^{\lambda}} \partial x.
\]
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Quod si jam primo capiamus $\lambda = 1$, quia ante invenimus pro eodem casu $x = \infty$ esse

$$\int \frac{x^{m-1}}{1-x^k} \, dx = \frac{\pi}{k \sin \frac{m \pi}{k}},$$

habebimus valorem istius integralis

$$\int \frac{x^{m-1}}{(1-x^k)^3} \, dx = \left(1 - \frac{m}{k}\right) \frac{\pi}{k \sin \frac{m \pi}{k}},$$

si quidem integrale etiam a termino $x = 0$ usque ad terminum $x = \infty$ extendatur.

§ 158. Quod si jam simili modo ponamus $\lambda = 2$, reperitur pro iisdem terminis integrationis

$$\int \frac{x^{m-1}}{(1-x^k)^3} \, dx = \left(1 - \frac{m}{k}\right) \left(1 - \frac{m}{2k}\right) \frac{\pi}{k \sin \frac{m \pi}{k}};$$

eodem modo si litterae $\lambda$ continuo maioribus valueres tribuantur, reperientur sequentes integralium formae omni attentione dignae

$$\int \frac{x^{m-1}}{(1-x^k)^4} \, dx = \left(1 - \frac{m}{k}\right) \left(1 - \frac{m}{2k}\right) \left(1 - \frac{m}{3k}\right) \frac{\pi}{k \sin \frac{m \pi}{k}};$$

$$\int \frac{x^{m-1}}{(1-x^k)^5} \, dx = \left(1 - \frac{m}{k}\right) \left(1 - \frac{m}{2k}\right) \left(1 - \frac{m}{3k}\right) \left(1 - \frac{m}{4k}\right) \frac{\pi}{k \sin \frac{m \pi}{k}};$$

$$\int \frac{x^{m-1}}{(1-x^k)^6} \, dx = \left(1 - \frac{m}{k}\right) \left(1 - \frac{m}{2k}\right) \left(1 - \frac{m}{3k}\right) \left(1 - \frac{m}{4k}\right) \left(1 - \frac{m}{5k}\right) \frac{\pi}{k \sin \frac{m \pi}{k}};$$

etc.

\[\text{etc.}\]

§ 159. Quare si littera $n$ denotet numerum quemcumque integrum, pro formula in titulo expressa, si ejus integrale $\alpha$ termino $x = 0$ usque ad $x = \infty$ extendatur, ejus valor sequenti modo se habebit
(1 - \frac{m}{k}) \left(1 - \frac{m}{2k}\right) \left(1 - \frac{m}{3k}\right) \left(1 - \frac{m}{4k}\right) \cdots \left(1 - \frac{m}{(n-1)k}\right) \frac{\pi}{k \sin \frac{m\pi}{k}}

qui ergo conveniet huic formulae integrali \[ \int \frac{x^{m-1} \partial x}{(1 + x^k)^n} \]

§. 160. Hic quidem necessario pro \(n\) ali i numeri præter integros accipi non licet: at vero per methodum interpolatlonum, quae fusius jam passim est explicata, hanc integrationem etiam ad casus, quibus exponens \(n\) est numerus fractus, extendere licet. Quod si enim sequentes formulae integrales a termino \(y = 0\) usque ad \(y = 1\) extendantur, in genere valor nostrae formulae propositae ita repræsentari poterit

\[ \int \frac{x^{m-1} \partial x}{(1 + x^k)^n} = \frac{\pi}{k \sin \frac{m\pi}{k}} \int y^{kn-m-1} \partial y \frac{(1-y^k)^{m-1}}{(1-y^k)^k} \]

Unde si fuerit \(m = 1\) et \(k = 2\), sequitur foro

\[ \int \frac{\partial x}{(1 + xx)^k} = \frac{\pi}{2} \int \frac{y^{n} \partial y}{\sqrt{1 - yy}} : \int \frac{\partial y}{\sqrt{1 - yy}} = \int \frac{y^{n} \partial y}{\sqrt{1 - yy}} \]

Ita si \(n = \frac{2}{3}\) erit.

\[ \int \frac{\partial x}{(1 + xx)^{\frac{3}{2}}} = \int \frac{y \partial y}{\sqrt{1 - yy}} \]

Iujus veritas sponte elucet, quia integrale prius generali est

\[ \frac{x}{\sqrt{1 + xx}} \]

posterior vero \(1 - \sqrt{1 - yy} \), quae facto \(x = \infty\)

et \(y = 1\), utique sunt aequalia. Caeterum pro hac integratione generali notasse juvabit, exponentem unitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum ex crescerent.
6) *Investigatio valoris integralis*

\[
\int \frac{x^{m-1} \partial x}{1 - 2x^k \cos \theta + x^{2k}}
\]

a termino \(x = 0\) usque ad \(x = \infty\) extensi. *Opus-cula analytica. Tom. II. Pag. 55 — 75.*

§ 161. Queramus primo integrale formulae propositae indefinietum, atque adeo omnes operationes ex primis analyseos principis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolvi nequit, sit in genere ejus factor duplicatus quicunque \(1 - 2x \cos \omega + xx\); evidens enim est, denominatorem fore productum ex \(k\) hujusmodi factoribus duplicatis. Cum igitur, posito hoc factorre \(= 0\), fiat \(x = \cos \omega \pm \sqrt{-1} \sin \omega\), eiam ipse denominator duplici modo evanescere debet, sive si ponatur

\[
\begin{align*}
x &= \cos \omega + \sqrt{-1} \sin \omega, \text{ sive} \\
x &= \cos \omega - \sqrt{-1} \sin \omega.
\end{align*}
\]

Constat autem omnes potestates harum formularum ita commodum exprimi posse, ut sit

\[
(\cos \omega \pm \sqrt{-1} \sin \omega)^k = \cos k\omega \pm \sqrt{-1} \sin k\omega,
\]

hinc igitur erit

\[
\begin{align*}
x^k &= \cos k\omega \pm \sqrt{-1} \sin k\omega \text{ et} \\
x^{2k} &= \cos 2k\omega \pm \sqrt{-1} \sin 2k\omega.
\end{align*}
\]

Substituamus ergo hos valores, et denominator noster evadet

\[
1 - 2 \cos \phi \cos k\omega \pm \cos 2k\omega \\
\pm 2 \sqrt{-1} \cos \phi \sin k\omega \pm \sqrt{-1} \sin 2k\omega.
\]

§ 162. Perspicuum igitur est hujus aequationis, tam terminos reales quam imaginarios seorsim se mutuo tollere de-
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bere, unde nascentur hae duae aequationes

I. \( 1 - 2 \cos \theta \cos k \omega + \cos 2k \omega = 0 \),

II. \( -2 \cos \theta \sin k \omega + \sin 2k \omega = 0 \).

Cum igitur sit

\[
\sin 2k \omega = 2 \sin k \omega \cos k \omega,
\]

posterior aequatio induet hanc formam:

\[
-2 \cos \theta \sin k \omega + 2 \sin k \omega \cos k \omega = 0,
\]

quae per \( 2 \sin k \omega \) divisa dat \( + \cos k \omega = \cos \theta \), idque

\[
\cos 2k \omega = \cos 2 \theta = \cos \theta^2 - \sin \theta^2 = 2 \cos \theta^2 - 1,
\]

qui valores in aequatione priorhe substitut praebent aequationem
identican, ita ut utique aequationi satisfiat sumendo \( \cos k \omega = \cos \theta \).

§. 463. Pro \( \omega \) igitur ejusmodi angulum assumi oportet,

ut fiat \( \cos k \omega = \cos \theta \), unde quidem statim deductur \( k \omega = \theta \),

ideoque \( \omega = \frac{\theta}{k} \). Verum quia infiniti sunt anguli cum

\( k \omega \) habentes, qui praeter ipsum angulum \( \theta \) sunt \( 2n \pi \pm \theta \),

\( 4n \pi \pm \theta \), \( 6n \pi \pm \theta \), etc. atque adeo in genere \( 2n \pi \pm \theta \), denotante

\( n \) omnès numeròs integros, quos nostra satisfiet, facturo

\( k \omega = 2n \pi \pm \theta \), unde colligitur angulus \( \omega = \frac{2n \pi \pm \theta}{k} \), sicque pro

\( \omega \) nascisceremur innumerables angulos satisfacientes, quorum aut

tem sufficiet tot assumisse, quot exponens \( k \) continet unitates;

successive igitur angulo \( \omega \) sequentes tribuamus valores

\[
\frac{\theta}{k}, \frac{2\pi + \theta}{k}, \frac{4\pi + \theta}{k}, \frac{6\pi + \theta}{k}, \ldots, \frac{2(k-1)\pi + \theta}{k}.
\]

Quodsi igitur angulo \( \omega \) successive singulos istos valores, quorum

numeros est \( = k \), tribuamus, formula \( 1 - 2x \cos \omega + xx \) omnès

suppeditabit factores duplicatos nostri denominatoris \( 1 - 2x^k \cos \theta \)

\( + x^{2k} \), quorum numeròs exit \( = k \).
SUPPLEMENTUM V.

§ 164. Invenis jam omnibus factoribus duplicatis nostri denominatoris, fractionem $\frac{x^{m-1}}{1 - 2x^k \cos \phi + x^{2k}}$ resolvi debet in tot fractiones partiales, quorum denominatores sint ipsi isti factores duplicati, quorum numeri sunt $k$, ita ut in genere tali fractioni partialis habitura sit talem formam $\frac{A + Bx}{1 - 2x \cos \omega + x^2}$, quam insuper resolvamus in binas simplices, etsi imaginarias, et cum sit

$$x^2 - 2x \cos \omega + 1 = (x - \cos \omega - \sqrt{-1} \sin \omega)(x - \cos \omega + \sqrt{-1} \sin \omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x - \cos \omega - \sqrt{-1} \sin \omega} + \frac{g}{x - \cos \omega + \sqrt{-1} \sin \omega},$$

ita ut totum resolutionis negotium huc redeat, ut ambo numeratores $f$ et $g$ determinentur; etsi enim invenis habetur summa ambarum fractionum

$$\frac{fx + gx - (f-g) \cos \omega + \sqrt{-1} (f+g) \sin \omega}{x^2 - 2x \cos \omega + 1},$$

ubi igitur erit

$$B = f + g \text{ et } A = (f - g) \sqrt{-1} \sin \omega - (f + g) \cos \omega.$$

§ 165. Per methodum igitur fractiones quascunque in fractiones simplices resolvendi statuamus

$$\frac{x^{m-1}}{1 - 2x^k \cos \phi + x^{2k}} = \frac{f}{x - \cos \omega - \sqrt{-1} \sin \omega} + R,$$

ubi $R$ complectatur omnes reliquas fractiones partiales. Hinc per $x - \cos \omega - \sqrt{-1} \sin \omega$ multiplicando habetur

$$\frac{x^m - x^{m-1} \cos \omega + \sqrt{-1} \sin \omega}{1 - 2x^k \cos \phi + x^{2k}} = f + R(x - \cos \omega - \sqrt{-1} \sin \omega),$$

quae aequatio cum vera esse debet, quicunque valor ipsi $x$ tribuatur, statuamus $x = \cos \omega + \sqrt{-1} \sin \omega$, ut membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra,
quia formula $x = \cos \omega = \sqrt{-1} \sin \omega$ simul est factor deno-
minatoris, facta hac substitutione tam numerator quam deno-
minator in nihilum abibunt, ut hinc nihil concludi posse
videatur.

§. 166. Hinc igitur utamur regula notissima, et loco
tam numeratris quam denominatris eorum differentialia scriba-
mus, unde nostra acquisitio accipiet sequentem formam

$$
\frac{m x^{m-1} - (m-1) x^{m-2} (\cos \omega + \sqrt{-1} \sin \omega)}{-2 k x^{k-1} \cos \theta + 2 k x^{2k-1}}
= \frac{m x^{m} - (m-1) x^{m-1} (\cos \omega + \sqrt{-1} \sin \omega)}{-2 k x^{k} \cos \theta + 2 k x^{2k}} = f,
$$
posito scilicet $x = \cos \omega + \sqrt{-1} \sin \omega$. Tum autem erit

$$
x^{m} = \cos m \omega + \sqrt{-1} \sin m \omega \text{ et }
$$

$$
x^{m-1} (\cos \omega + \sqrt{-1} \sin \omega) = x^{m} = \cos m \omega + \sqrt{-1} \sin m \omega,
$$
et pro denominatore

$$
x^{k} = \cos k \omega + \sqrt{-1} \sin k \omega \text{ et }
$$

$$
x^{2k} = \cos 2 k \omega + \sqrt{-1} \sin 2 k \omega;
$$
unde sit numerator,

$$
x^{m} = \cos m \omega + \sqrt{-1} \sin m \omega
$$
et denominator

$$
-2 k \cos \theta \cos k \omega + 2 k \cos 2 k \omega,
$$

$$
-2 k \sqrt{-1} \cos \theta \sin k \omega + 2 k \sqrt{-1} \sin 2 k \omega.
$$

§. 167. Pro denominatore reducendo recordemur, jam
supra inventum esse $\cos k \omega = \cos \theta$, unde sit $\sin k \omega = \sin \theta$,
tum vero

$$
\cos 2 k \omega = \cos 2 \theta = 2 \cos \theta^2 - 1 \text{ et }
$$

$$
\sin 2 k \omega = \sin 2 \theta = 2 \sin \theta \cos \theta,
$$

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quibus valoribus adhibitis denominator noster erit

\[ 2k \cos \theta \cos \theta - 2k + 2k \sqrt{1 - \sin \theta \cos \theta} = 2k \sin \theta \cos \theta - 2k \sin \theta \cos \theta \]

quamobrem hoc valore adhibito habebimus.

\[ f = \frac{\cos m \omega + \sqrt{1 - \sin m \omega}}{2k \sin \theta (\sqrt{1 - \cos \theta} - \sin \theta)}. \]

Simul vero hinc sine novo calculo deducemus valorem \( g \), quippe qui ab \( f \) ratione signi \( \sqrt{-1} \) tantum discrepat, sicque erit

\[ g = \frac{\cos m \omega - \sqrt{1 - \sin m \omega}}{-2k \sin \theta (\sin \theta \cdot \sqrt{1 - \cos \theta})}. \]

§ 468. Inventis autem his litteris \( f \) et \( g \), pro litteris \( A \) et \( B \) colligemus primo.

\[ f + g = \frac{\cos \theta \sin m \omega - \sin \theta \cos m \omega}{k \sin \theta} = \frac{\sin (m \omega - \theta)}{k \sin \theta}. \]

tum vero erit

\[ f - g = -\frac{\sqrt{-1} \cos (m \omega - \theta)}{k \sin \theta}. \]

Ex his igitur reperiemus

\[ B = \frac{\sin (m \omega - \theta)}{k \sin \theta} \]

\[ A = \frac{\sin \theta \cos (m \omega - \theta) - \cos \theta \sin (m \omega - \theta)}{k \sin \theta} = \frac{\sin [(m \omega - \theta) - \omega]}{k \sin \theta}. \]

ubi erno imaginaria sponte se mutuo destruxerunt.

§ 469. Inventis his valoribus \( A \) et \( B \), investigari oportet integrale partiale \( \int_{x=1}^{x=\frac{\pi}{2}} (A + B \cos \omega) \theta x \). ubi, cum denominatoris differentiale sit

\[ 2x \partial x - 2 \partial x \cos \omega = 2 \partial x (x - \cos \omega), \]

statuamus.

\[ A - B x = B (x - \cos \omega) + C, \]

eritque

\[ C = A - B \cos \omega, \]
Hinc igitur erit
\[ C = \frac{\cos \omega \sin (m\omega - \theta) - \sin [(m\omega - \theta) - \omega]}{h \sin \theta} \]

Quia vero
\[ -\sin (m\omega - \theta - \omega) = -\sin (m\omega - \theta) \cos \omega + \cos (m\omega - \theta) \sin \omega, \]
\[ C = \frac{\sin \omega \cos (m\omega - \theta)}{h \sin \theta}. \]

Hac ergo forma adhibita, formula integranda \( \frac{(A + Bx) \partial x}{1 - 2x \cos \omega + x^2} \) discerpatur in duas partes
\[ \frac{B(x - \cos \omega) \partial x}{1 - 2x \cos \omega + x^2} + \frac{C \partial x}{1 - 2x \cos \omega + x^2}. \]

Hic igitur prioris partis integrale manifesto est
\[ B f \sqrt{(1 - 2x \cos \omega + x^2)}, \]
alterius vero partis facile patet integrale per arcum circuli ex pressum iri, cujus tangens sit \( \frac{x \sin \omega}{1 - x \cos \omega} \). Ad hoc integrale inventium ponamus
\[ \int \frac{C \partial x}{1 - 2x \cos \omega + x^2} = \mathcal{D} \cdot \text{Arc. tang.} \frac{x \sin \omega}{1 - x \cos \omega}, \]
et summis differentialibus, quia \( \partial \), Arc. tang. \( t \) aequale est \( \frac{\partial t}{1 + t^2} \), habeimus
\[ \frac{C \partial x}{1 - 2x \cos \omega + x^2} = \mathcal{D} \cdot \frac{\partial x \sin \omega}{1 - 2x \cos \omega + x^2}. \]
unde manifesto fit
\[ \mathcal{D} = \frac{C}{\sin \omega} = \frac{\cos (m\omega - \theta)}{h \sin \theta}. \]

§. 170. Substituamus igitur loco \( B \) et \( \mathcal{D} \) valores modo inventos, et ex singulis factoribus denominatoris
\[ 1 - 2x^k \cos \theta + x^{2k}, \]
quorum forma est \( 1 - 2x \cos \omega + x^2 \), ortur pars integralis constans ex membro logarithmico et arcu circulari, quae erit
\[ \frac{\sin (m\omega - \theta)}{h \sin \theta} \int \sqrt{(1 - 2x \cos \omega + x^2)} + \frac{\cos (m\omega - \theta)}{h \sin \theta} \text{Arc. tang.} \frac{x \sin \omega}{1 - x \cos \omega}. \]
SUPPLEMENTUM V.

quae evanesco sumto $x = 0$. In hac igitur forma tantum opus. est, ut loco $\omega$ successive scribamus valores supra indicatos, sollicit

$$\omega = \frac{\theta}{k}, \frac{2\pi + \theta}{k}, \frac{3\pi + \theta}{k}, \frac{6\pi + \theta}{k}, \text{ etc.}$$
donee perveniatur ad $\frac{2(k-1)\pi + \theta}{k}$; tum enim summa omnium harum formarum praebet totum integrale indefinitum formulae propositae.

§. 171. Postquam igitur integrale indefinitum elicuimus, nihil alium superest, nisi ut in eo faciamus $x = \infty$, quo facto

$$\sqrt{(1 - 2x \cos \omega + x^2)} = x - \cos \omega,$$

erit $Bl(x - \cos \omega)$. Est vero

$$l(x - \cos \omega) = l x - \frac{\cos \omega}{x} = l x, \text{ ob } \frac{\cos \omega}{k} = \theta,$$
quamobrem facto $x = \infty$ quaelibet pars logarithmica habebit hanc
formam $\frac{\sin(\frac{(m\omega - \theta)}{k})}{\sin \frac{\theta}{k}} l x$. Deinde pro partibus a circulo pendentibus, facto $x = \infty$ fit

$$\frac{x \sin \omega}{1 - x \cos \omega} = \tang \omega = \tang(\pi - \omega),$$
sicque arccus, cujus hac est tangens, erit $\pi - \omega$, hincque pars

$$\frac{\cos(\frac{(m\omega - \theta)}{k})}{\cos \frac{\theta}{k}} (\pi - \omega),$$
sicque quaecunque fiant.

§. 172. Cum quilibet valor anguli $\omega$ in genere hanc

habeat formam $\frac{2\sin \omega + \theta}{k}$; erit angulus

$$m \omega - \theta = \frac{2\pi \sin \omega - \theta (k - m)}{k} \text{ et } \pi - \omega = \frac{\pi (k - 2\pi - \theta)}{k}.$$

Ponamus brevissima gratia

$$\frac{\theta (k - m)}{k} = \zeta \text{ et } \frac{m \pi}{k} = \alpha, \text{ ut sit } m \omega - \theta = 2\zeta \alpha - \frac{\pi}{k},$$
ubi loco $i$ scribi debent successive numeri $0, 1, 2, 3, \text{ etc. usque } ad \ k - 1$. Hinc igitur si omnes partes logarithmicas in unam.
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summam colligamus, ea ita repreaesentari poterit

\[ \frac{1^2}{k \sin \theta} \left[ -\sin \zeta + \sin (2\alpha - \zeta) + \sin (4\alpha - \zeta) + \sin (6\alpha - \zeta) \right. \\
\left. + \sin (8\alpha - \zeta) + \cdots + \sin \left[ 2(k - 1)\alpha - \zeta \right] \right]; \]

ubi quidem ex ilis, quae hactenus sunt tradita, facile suspicari licet, 
totam hanc progressionem ad nihilum redigi. Verum hoc ipsum
firma demonstratione muniri necesse est.

§. 173. Ad hoc ostendendum ponamus

\[ S = -\sin \zeta + \sin (2\alpha - \zeta) + \sin (4\alpha - \zeta) + \cdots + \sin \left[ 2(k - 1)\alpha - \zeta \right], \]

multiplicemus utrinque per 2 sin \( \alpha \), et cum sit

\[ 2 \sin \alpha \sin \Phi = \cos (\alpha - \Phi) - \cos (\alpha + \Phi), \]

hujus reductionis ope obtinebimus sequentem expressionem:

\[ 2S \sin \alpha = \cos (\alpha + \zeta) + \cos (\alpha - \zeta) + \cos (3\alpha - \zeta) + \cos (5\alpha - \zeta) \]
\[ - \cos (\alpha - \zeta) - \cos (3\alpha - \zeta) - \cos (5\alpha - \zeta) \]
\[ + \cos [(2k - 3)\alpha - \zeta] - \cos [(2k - 1)\alpha - \zeta], \]
\[ \cdots - \cos [(2k - 3)\alpha - \zeta] \]

unde deletis terminis se mutue destruentibus habebitur:

\[ 2S \sin \alpha = \cos (\alpha + \zeta) - \cos \left[ (2k - 1)\alpha - \zeta \right]. \]

§. 174. Ponamus hos duos angulos, qui sunt relictii,
\[ \alpha + \zeta = p \] et \( (2k - 1)\alpha - \zeta = q \), eritque eorum summa \( p + q = 2\alpha k \). Quia vero est \( \alpha = \frac{m\pi}{k} \), erit \( p + q = 2m\pi \); hoc est

mulplo totias circuli peripheriae, ob \( m \) numerum integrum. Quare

cum sit \( q = 2m\pi - p \), erit \( \cos q = \cos p \); unde patet summam

inventam nihil esse aequalem, sicque manifestum est, omnes par-
tes logarithmicas, quae in integrale formulae nostrae ingrediuntur,
casu \( x = \infty \) se mutuo destruer.
SUPPLEMENTUM V.

§ 175. Progrediamur igitur ad partes circulares, quorum forma generalis, ut vidimus, est \( \frac{\cos\left(\frac{m\alpha - \xi}{k}\right)}{k \sin\theta} \) \( (\pi - \omega) \), quae
\[
\frac{\cos\left(2i\alpha - \xi\right)}{k \sin\theta} \left(\pi - \frac{2im - i}{k}\right) = \frac{\cos\left(2i\alpha - \xi\right)}{k \sin\theta} \left(\pi - \frac{2im}{k} - \frac{\theta}{n}\right).
\]
Hic ponatur porro \( \frac{\pi}{k} = \beta \) et \( \pi - \frac{\theta}{n} = \gamma \), ut forma generalis sit \( \frac{\cos\left(2i\alpha - \xi\right)}{k \sin\theta} \) \( (\gamma - 2i\beta) \). Quare si dico 1 scribas ordine valores, \( 0, 1, 2, 3, 4 \), usque ad \( k - 1 \), omnes partes circulares hane progressionem censitent
\[
\frac{1}{k \sin\theta} [\gamma \cos\xi + (\gamma - 2\beta) \cos\left(2\alpha - \xi\right) + (\gamma - 4\beta) \cos\left(4\alpha - \xi\right) + \ldots + \left(\gamma - 2\left(k - 1\right)\beta\right) \cos\left[2\left(k - 1\right)\alpha - \xi\right]].
\]

Ponamus igitur
\[
S = \gamma \cos\xi + (\gamma - 2\beta) \cos\left(2\alpha - \xi\right) + (\gamma - 4\beta) \cos\left(4\alpha - \xi\right) + \ldots + \left(\gamma - 2\left(k - 1\right)\beta\right) \cos\left[2\left(k - 1\right)\alpha - \xi\right]
\]
\[= \frac{S}{k \sin\theta}, \] quae ergo praebet valorem quasitum formulae integralis propositae, casu quo post integracionem statitur \( \alpha = \infty \), ita ut totum negotium in investigando valore ipsius \( S \) versetur.

§ 175. Hunc in finem multiplicemus utrinque per \( 2 \sin\alpha \), et cum in genere sit
\[2 \sin\alpha \cos\phi = \sin\left(\alpha + \phi\right) - \sin\left(\phi - \alpha\right),\]
hac reductione in singulis terminis facta, perveniemus ad hanc aequationem
\[
2S \sin\alpha = \gamma \sin\left(\alpha + \xi\right) + \gamma \sin\left(\alpha - \xi\right) + (\gamma - 2\beta) \sin\left(3\alpha - \xi\right) - (\gamma - 2\beta) \sin\left(\alpha - \xi\right) - (\gamma - 4\beta) \sin\left(3\alpha - \xi\right) + (\gamma - 4\beta) \sin\left(5\alpha - \xi\right) + \ldots + [\gamma - 2\left(k - 1\right)\beta] \sin\left[2\left(k - 1\right)\alpha - \xi\right]
- (\gamma - 6\beta) \sin\left(5\alpha - \xi\right)
\]
ubi praeter primum et ultimum terminum omnes reliqui con-
trahi possunt, ëta ut prodecat
\[2\sin \alpha = \gamma \sin (\alpha + \zeta) + 2\beta \sin (\alpha - \zeta) + 2\beta \sin (3\alpha - \zeta) + 2\beta \sin (5\alpha - \zeta) + \ldots + 2\beta \sin [(2k-3)\alpha - \zeta] + [\gamma - \zeta (k-1)]\beta \sin [(2k-4)\alpha - \zeta].\]

§. 477. Jam pro hac serie summam saepe possumus porro
\[T = 2\sin (\alpha - \zeta) + 2\sin (3\alpha - \zeta) + 2\sin (5\alpha - \zeta) + \ldots + 2\sin [(2k-3)\alpha - \zeta].\]

ut habeamus
\[2\sin \alpha = \gamma \sin (\alpha + \zeta) + [\gamma - 2(k-1)\beta] \sin [(2k-1)\alpha - \zeta] + \beta T.\]

Jam multiplicemus, ut hactenus, per sin. \(\alpha\), et cum sit
\[2\sin \alpha \sin (\Phi - \alpha) = \cos (\Phi - \alpha) - \cos (\Phi + \alpha),\]

facta haec reductione nanciscimus
\[T \sin \alpha = \cos \zeta + \cos (2\alpha - \zeta) + \cos (4\alpha - \zeta) + \ldots + \cos [2(k-2)\alpha - \zeta] - \cos (2\alpha - \zeta) - \cos (4\alpha - \zeta) - \ldots - \cos [(2k-2)\alpha - \zeta].\]

unde deletis terminis, quae se mutuo destruunt, remanebit tantum
ista expressio.

\[T \sin \alpha = \cos \zeta + \cos [2(k-1)\alpha - \zeta].\]

Cum igitur sit \(\alpha = \frac{m\pi}{k}\), erit
\[2(k-1)\alpha = 2m \pi - \frac{2m\pi}{k},\]
aquis loco scribere licet \(\frac{2m\pi}{k}\), unde \(\zeta = \frac{\theta (k-m)}{k}\), erit
\[T \sin \alpha = \cos \frac{2(k-m)\pi}{k} \cos \left(\frac{2m\pi - \theta (k-m)}{k}\right),\]

§. 478. Nunc vero notetur in genere esse
\[\cos p = \cos q = 2 \sin \frac{q+p}{2} \sin \frac{q-p}{2},\]
quare cum sit
\[p = \frac{\theta (k-m)}{k} \text{et} q = \frac{2m\pi - \theta (k-m)}{k}, \text{erit}\]
\[\frac{q+p}{2} = \frac{m\pi + \theta (k-m)}{k} \text{et} \frac{q-p}{2} = \frac{m\pi}{k}.\]
unde sequitur fore

\[ T \sin \alpha = 2 \sin \left( \frac{m \pi + \theta (k - m)}{k} \right) \sin \frac{m \pi}{k}, \]

ideoque

\[ T = 2 \sin \left( \frac{m \pi + \theta (k - m)}{k} \right), \text{ ob } \alpha = \frac{m \pi}{k}. \]

§ 179. Hoc igitur valorem \( T \) invento reperiemus porro

\[ 2 S \sin \alpha = \left( \gamma + 2 \beta \right) \sin \left( \alpha + \zeta \right) + \left[ \gamma - 2 (k - 1) \beta \right] \sin \left[ (2k - 1) \alpha - \zeta \right] \]

\[ + 2 \beta \sin \left( \frac{m \pi + \theta (k - m)}{k} \right), \]

quae ob \[ \frac{m \pi + \theta (k - m)}{k} = \alpha \pm \zeta \]

reductur ad hanc formam

\[ 2 S \sin \alpha = (\gamma + 2 \beta) \sin \left( \alpha + \zeta \right) + (\gamma - 2 (k - 1) \beta) \sin \left[ (2k - 1) \alpha - \zeta \right], \]

qua ita representari potest

\[ 2 S \sin \alpha = (\gamma + 2 \beta) \left[ \sin \left( \alpha + \zeta \right) + \sin \left[ (2k - 1) \alpha - \zeta \right] \right] \]

\[ - 2 \beta k \sin \left[ (2k - 1) \alpha - \zeta \right], \]

ubi pro parte priore, ob

\[ \sin \frac{p + q}{2} = 2 \sin \frac{p - q}{2} \cos \frac{p - q}{2}, \text{ erit} \]

\[ \frac{p + q}{2} = \alpha k \text{ et } \frac{p - q}{2} = (k - 1) \alpha - \zeta, \]

unde pars ipsa prior fit

\[ 2 \left( \gamma + 2 \beta \right) \sin \alpha k \cos \left[ (k - 1) \alpha - \zeta \right], \]

ubi cum sit \( \alpha k = m \pi \), erit \( \sin \alpha k = 0 \), ita ut tantum supersit

\[ 2 S \sin \alpha = -2 \beta k \sin \left[ (2k - 1) \alpha - \zeta \right], \]

hincque

\[ S = -\frac{\beta k \sin \left[ (2k - 1) \alpha - \zeta \right]}{\sin \alpha}. \]

Est vero

\[ (2k - 1) \alpha - \zeta = \frac{m \pi}{k} - \frac{\theta (k - m)}{k}. \]
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omisso igitur termino \(2m\pi\), exit

\[
S = -\pi \sin \left[ \frac{\pi + \delta (k-m)}{k} \right] \frac{m\pi}{\sin \frac{m\pi}{k}}\]

ideoque valor quaesitus

\[
S = -\pi \sin \left[ \frac{\pi + \delta (k-m)}{k} \right] \frac{m\pi}{k \sin \delta \sin \frac{m\pi}{k}}
\]

quae forma reducitur ad hanc

\[
\frac{\pi \sin \left[ \frac{\pi - \delta}{k} + \frac{\delta k}{k} \right]}{k \sin \delta \sin \frac{m\pi}{k}}
\]

§. 180. Contemplemur hic ante omnia casum quo \(\delta = \frac{\pi}{2}\), et formula integralis proposita abit in hanc \(\int \frac{x^{m-1}}{1 + x^k} \, dx\), cujus

ergo valor, si post integrationem ponatur \(x = \infty\), evadet

\[
\frac{\pi \sin \left( \frac{\pi}{2} + \frac{m\pi}{2k} \right)}{k \sin \frac{m\pi}{k}} = \frac{\pi \cos \frac{m\pi}{2k}}{k \sin \frac{m\pi}{k}}
\]

Quia igitur est

\[
\sin \frac{m\pi}{k} = 2 \sin \frac{m\pi}{2k} \cos \frac{m\pi}{2k}
\]

prohibit iste valor \(\frac{\pi}{2k \sin \frac{m\pi}{k}}\), qui valor egregie convenit cum

cu, quem non ita pridem pro formula \(\int \frac{x^{m-1}}{1 + x^k} \, dx\) assignavimus, si

quidem loco \(k\) scribatur \(2k\).

§. 181. Evolvamus etiam casum quo \(\delta = \pi\), et for-

mula nostra integralis \(\int \frac{x^{m-1}}{(1 + x^k)^2} \, dx\), cujus ergo, facto \(x = \infty\),

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valor erit\[\frac{\pi \sin \left[\frac{m(\pi - \theta) + \theta}{k}\right]}{k \sin \theta \sin \frac{m\pi}{k}} = -\frac{\pi}{k} \sin \left[\frac{m(\pi - \theta) + \theta}{k}\right] \sin \theta}\]

Hujus autem posterioris fractionis, casu \(\theta = \pi\), tam numerator quam denominator evanescebit; quare, ut ejus verum valor eruat, loco utiusque ejus differentiale scribamus, quo facto ista fractio abibit in hanc\[\frac{\partial}{\partial \theta} \left(1 - \frac{m}{k}\right) \cos \left[\frac{m(\pi - \theta) + \theta}{k}\right],\]
cujus valor facto \(\theta = \pi\) nunc manifesto est \(1 - \frac{m}{k}\); sicque valor integralis quaesitus erit \(\left(1 - \frac{m}{k}\right) \frac{\pi}{k \sin \frac{m\pi}{k}}\), prorsus uti in superiore dissertatione invenimus.

§ 182. Quo autem valorem generalis inventum commodiorem reddamus, ponamus \(\pi - \theta = \eta\), fietque sinus \(\theta = \sin \eta\) et cos \(\theta = \cos \eta\); tum vero erit angulus\[\frac{m(\pi - \theta)}{k} + \theta = \frac{m\eta}{k} + \pi - \eta,\]
cujus sinus est \(\sin \left(1 - \frac{m}{k}\right) \eta\), unde valor quaesitus nostrae formulae erit \(\frac{\pi \sin \left(1 - \frac{m}{k}\right) \eta}{k \sin \eta \sin \frac{m\pi}{k}}\), atque hinc tandem sequens adepti sumus theorema.

Theorem.

§ 183. Si haec formula integralis\[\int \frac{x^{m-1} \partial x}{1 + 2x^k \cos \eta + x^k}\]
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a termino \( x = 0 \) usque ad terminum \( x = \infty \) extendatur, ejus
valor erit
\[
\frac{\pi \sin \left( 1 - \frac{m}{k} \right) \eta}{k \sin \eta \sin \frac{m\pi}{k}}, \text{ sive cum sit }
\sin \left( 1 - \frac{m}{k} \right) \eta = \sin \eta \cos \frac{m\eta}{k} - \cos \eta \sin \frac{m\eta}{k},
\]
iste valor etiam hoc modo exprimi potest
\[
\frac{\pi \cos \frac{m\eta}{k}}{k \sin \frac{m\eta}{k}} - \frac{\pi \sin \frac{m\eta}{k}}{k \tan \eta \sin \frac{m\pi}{k}}.
\]

§. 184. Consideremos nunc alio modo hanc formulam
integralem
\[
\int \frac{x^{m-1} \, \partial x}{1 + 2x^k \cos \eta + x^{2k}},
\]
cujus valor a termino \( x = 0 \) usque ad \( x = 1 \) ponatur \( P \),
ejusdem vero valor ab \( x = 1 \) usque ad \( x = \infty \) ponatur \( Q \),
ita ut \( P + Q \) exhibere debeat ipsum valorem ante inventum.
Nunc vero pro valore \( Q \) inveniendó ponamus \( x = \frac{1}{y} \), et formula
nostra ita representata
\[
\frac{y^m}{1 + 2y^{-k} \cos \eta + y^{-2k}} \cdot \frac{\partial x}{x},
\]
ob \( \frac{\partial x}{x} = \frac{\partial y}{y} \) inducet hanc formam
\[
- \int \frac{y^{-m}}{1 + 2y^{-k} \cos \eta + y^{-2k}} \cdot \frac{\partial y}{y} = - \int \frac{y^{2k-m-1} \, \partial y}{y^{2k} + 2y^k \cos \eta + 1},
\]
cujus valor a termino \( y = 1 \) usque ad \( y = 0 \) extendi debet.
Commutatis igitur his terminis habebimus
\[
Q = + \int \frac{y^{2k-m-1} \, \partial y}{y^{2k} + 2y^k \cos \eta + 1},
\]
a termino \( y = 0 \) usque ad \( y = 1 \).
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§. 186. Quia in utraque forma pro \( P \) et \( Q \) eadem conditio integrationis praeoribitur, a termino 0 usque ad 1, nihil impedit quo minus in posteriore loco \( y \) scribamus \( x \), unde pro \( P + Q \) habeimus hanc formam integralem

\[
\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos \eta + x^{2k}} \, dx,
\]

cujus valor, a termino \( x = 0 \) usque ad \( x = 1 \) extensus, aequabitur huic expressioni

\[
\frac{\pi \sin \left( \frac{1 - m}{k} \right) \eta}{k \sin \eta \sin \frac{\pi m}{k}}.
\]

Comparatis igitur his binis formulis integralibus nanciscemur sequens theorema notatu maxime dignum.

Theorema.

§. 186. Haec formula integralis

\[
\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos \eta + x^{2k}} \, dx,
\]

a termino \( x = 0 \) usque ad terminum \( x = 1 \) extensa, aequalis est huic formulae integrali

\[
\int \frac{x^{m-1}}{1 + 2x^k \cos \eta + x^{2k}} \, dx,
\]

a termino \( x = 0 \) usque ad terminum \( x = \infty \) extensae: utriusque enim valor erit

\[
\frac{\pi \sin \left( \frac{1 - m}{k} \right) \eta}{k \sin \eta \sin \frac{\pi m}{k}}.
\]

§. 187. Quod si hanc fractionem

\[
\sin \eta
\]

in seriem infinitam evolvamus, quae sit

\[
\sin \eta = Ax^k + Bx^{2k} + Cx^{3k} + Dx^{4k} + Ex^{5k} + \text{etc.}
\]
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per denominatorem multiplicando perveniemus ad hanc expressionem infinitam

\[
\sin \gamma = \sin \eta + A x^k + B x^{2k} + C x^{3k} + D x^{4k} + E x^{5k} + F x^{6k} + \text{etc.}
\]

\[
+ 2 \sin \eta \cos \eta + 2A \cos \eta + 2B \cos \eta + 2C \cos \eta + 2D \cos \eta + 2E \cos \eta + \text{etc.}
\]

\[
+ \sin \eta + A + B + C + D + \text{etc.}
\]

unde singulis terminis ad nihilum reductis reperiemus

1°. \( A = 2 \sin \eta \cos \eta = 0 \), hincque \( A = -\sin 2\eta \)

2°. \( B = 2A \cos \eta + \sin \eta = 0 \), unde fit \( B = \sin 3\eta \)

3°. \( C = 2B \cos \eta + A = 0 \), unde fit \( C = -\sin 4\eta \)

4°. \( D = 2C \cos \eta + B = 0 \), unde fit \( D = \sin 5\eta \)

\[\text{etc.}\]

\[\text{etc.}\]

ita ut nostra fraction \( \frac{\sin \eta}{1 + 2 x^k \cos \eta - x^{2k}} \) resolvatur in hanc seriem:

\[
\sin \eta - x^k \sin 2\eta - x^{2k} \sin 3\eta - x^{3k} \sin 4\eta - x^{4k} \sin 5\eta - \text{etc.}
\]

§ 188. Multiplicemus nunc hanc seriem per

\[
x^{m-1} \frac{\partial x}{1 + 2 x^k \cos \eta - x^{2k}} \partial x,
\]

et post integrationem faciamus \( x = 1 \), ut obtineamus valorem hujus formulæ:

\[
\sin \eta \int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2 x^k \cos \eta - x^{2k}} \partial x.
\]

pro casu \( x = 1 \), hocque modo perveniemus ad geminas sequentes series

\[\frac{\sin \eta}{m} + \frac{\sin 2\eta}{m+k} + \frac{\sin 3\eta}{m+2k} + \frac{\sin 4\eta}{m+3k} + \frac{\sin 5\eta}{m+4k} + \text{etc.}\]

\[\frac{\sin \eta}{2k-m} + \frac{\sin 2\eta}{8k-m} + \frac{\sin 3\eta}{4k-m} + \frac{\sin 4\eta}{5k-m} + \frac{\sin 5\eta}{6k-m} + \text{etc.}\]

Aggregatum igitur harum duarum serierum junctim summaturum

\[\text{summarum}\]
aequabitur huic valori \( \frac{\pi \sin \left(1 - \frac{m}{k} \right) \eta}{k \sin \frac{m\pi}{k}} \), unde subjungamus aðhue
istud theorema.

Theorema.

§ 189. Si \( \eta \) denotet angulum quemcunque, litterae vero
\( m \) et \( k \) pro lubitu accipientur, ex eisque binae sequentes series
formentur

\[
P = \frac{\sin \frac{\eta}{m}}{\frac{\sin 2\eta}{m+k}} - \frac{\sin 3\eta}{m+2k} - \frac{\sin 4\eta}{m+3k} + \frac{\sin 5\eta}{m+4k} - \text{etc.}
\]

\[
Q = \frac{\sin \frac{\eta}{2k-m}}{\frac{\sin 2\eta}{3k-m}} + \frac{\sin 3\eta}{4k-m} - \frac{\sin 4\eta}{5k-m} + \frac{\sin 5\eta}{6k-m} - \text{etc.}
\]

neutrius quidem summa exhiberi potest, utriusque autem junctim
sumtac summa erit

\[
P + Q = \frac{\pi \sin \left(1 - \frac{m}{k} \right) \eta}{k \sin \frac{m\pi}{k}}.
\]

Corollarium.

§ 190. Quod si ergo angulum \( \eta \) infinite parvum capiamus, ut fiat
\[
\sin \frac{\eta}{\eta} = \sin 2\eta = 2\eta, \sin 3\eta = 3\eta, \text{etc.}
\]

quia in formula summae sint

\[
\sin \left(1 - \frac{m}{k} \right) \eta = \left(1 - \frac{m}{k} \right) \eta;
\]

si utrinque per \( \eta \) dividamus, obtinebimus sequentem seriem geminatam

\[
\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}
\]

\[
\frac{1}{2k-m} - \frac{2}{3k-m} + \frac{3}{4k-m} - \frac{4}{5k-m} + \frac{5}{6k-m} - \text{etc.}
\]

cujus ergo summa erit \( \left(1 - \frac{m}{k} \right) \frac{\pi}{k \sin \frac{m\pi}{k}} \), ubi notetur, ambas istas
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series non incongrue in hanc simplicem contrahi posse

\[ \frac{2k}{m(2k-m)} - \frac{2k}{(m+k)(3k-m)} - \frac{18k}{(m+3k)(6k-m)} - \frac{92k}{(m+8k)(9k-m)} - etc. \]

ubi numeratores sunt numeri quadrati duplicati.

§. 191. Formulae autem, quorum valores hactenus invencionis, multo concinnius et elegantius exprimere possunt, si loco exponens \( m \) scribamus \( k - n \), tum enim in valore integrali invento fiet \( (1 - \frac{n}{k}) \eta = \frac{n\eta}{k} \); at vero pro denominatore fiet \( \frac{n\pi}{k} = \pi - \frac{n\pi}{k} \), cujus sinus esset \( \frac{n\pi}{k} \); sicque nostra formula inventa hanc induet formam

\[ \frac{\pi \sin \frac{n\eta}{k}}{k \sin \eta \sin \frac{n\pi}{k}} \]

quae ergo exprimet valorem hujus formulae integralis

\[ \int \frac{x^{k-n-1} \partial x}{1 + 2x^k \cos \eta + x^{2k}} \]

ab \( x = 0 \) usque ad \( x = \infty \), ut et hujus formule

\[ \int \frac{x^{k-n-1} + x^{k-n} \partial x}{1 + 2x^k \cos \eta + x^{2k}} \]

a termino \( x = 0 \) usque ad terminum \( x = 1 \); et quia utriusque valor est \( \frac{\pi \sin \frac{n\eta}{k}}{k \sin \eta \sin \frac{n\pi}{k}} \); perspicuum est eum manere eundem, etsi loco \( n \) scribatur \( - n \), ex quo prior formula ita representari poterit

\[ \int \frac{x^{k+n-1} \partial x}{1 + x^k \cos \eta + x^{2k}} \]

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.
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§. 192. Ponendo \( m = k - n \) etiam series nostra geminata pulchriorëm accipiet faciem; habebitur enim

\[
\frac{\sin \frac{\eta}{k-n}}{k+n} - \frac{\sin \frac{2\eta}{2k-n}}{2k+n} + \frac{\sin \frac{3\eta}{3k-n}}{3k+n} - \frac{\sin \frac{4\eta}{4k-n}}{4k+n} + \text{etc.}
\]

\[
\frac{\sin \frac{\eta}{k-n}}{k+n} - \frac{\sin \frac{2\eta}{2k-n}}{2k+n} + \frac{\sin \frac{3\eta}{3k-n}}{3k+n} - \frac{\sin \frac{4\eta}{4k-n}}{4k+n} + \text{etc.}
\]

cujus ergo summa erit \( \frac{\pi \sin \frac{n\eta}{k}}{k} \). Tum vero si hae geminæ series in unam contrahantur, et utrinque per \( 2k \) dividatur, obtinebitur sequens summatio memoratu digna

\[
\frac{\pi \sin \frac{n\eta}{k}}{2kk \sin \frac{n\pi}{k}} = \frac{\sin \frac{\eta}{kk-nn}}{kk-nn} - \frac{2 \sin \frac{2\eta}{4kk-nn}}{4kk-nn} + \frac{3 \sin \frac{3\eta}{9kk-nn}}{9kk-nn} - \frac{4 \sin \frac{4\eta}{16kk-nn}}{16kk-nn} + \text{etc.}
\]

§. 193. Quodsi haec postrema series differentietur, sumendo solum angulum \( \eta \) variabilem, ob

\[
\sin \frac{n\eta}{k} = \frac{n \partial \eta}{k} \cos \frac{\eta}{k}
\]

habebimus.

\[
\frac{\pi n \cos \frac{n\eta}{k}}{2k^3 \sin \frac{n\pi}{k}} = \frac{\cos \frac{\eta}{kk-nn}}{kk-nn} - \frac{4 \cos \frac{2\eta}{4kk-nn}}{4kk-nn} + \frac{9 \cos \frac{3\eta}{9kk-nn}}{9kk-nn} - \frac{16 \cos \frac{4\eta}{16kk-nn}}{16kk-nn} + \text{etc.}
\]

Unde si sumatur \( \eta = 0 \), orietur ista summatio

\[
\frac{\pi n}{2k^3 \sin \frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}
\]

Sin autem sumatur \( \eta = 90^\circ = \frac{\pi}{2} \), erit

\[
\cos \frac{\eta}{kk-nn} = 0, \cos \frac{2\eta}{4kk-nn} = -1, \cos \frac{3\eta}{9kk-nn} = 0, \cos \frac{4\eta}{16kk-nn} = +1 \text{ etc.}
\]

unde nascitur sequens series

\[
\frac{n \pi \cos \frac{n\pi}{2k}}{2k^3 \sin \frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}
\]
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Quia autem \( \frac{n\pi}{k} = 2 \sin \frac{n\pi}{2k} \cos \frac{n\pi}{2k} \), erit ejusdem seriei summa

\[
\frac{n\pi}{4k^3 \sin \frac{n\pi}{2k}}
\]

§ 194. At si series illa § 192. exhibita in \( \partial \gamma \) ducatur et integretur, \( \varepsilon \) \( \varepsilon \)

\[
\sin \frac{n\pi}{k} = -\frac{k}{n} \cos \frac{n\pi}{k}, \text{ cr\`at.}
\]

\[
C = \frac{\pi \cos \frac{n\pi}{k}}{2nk \sin \frac{n\pi}{k}} = -\cos \eta + \cos 2\eta - \cos 3\eta + \cos 4\eta + \ldots
\]

\[
= \frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \frac{1}{16kk-nn} + \ldots
\]

Ut autem hic constantem addendum \( C \) definiamus, sumamus \( \eta = 0 \), fietque

\[
C = \frac{\pi}{2nk \sin \frac{n\pi}{k}} = \frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \ldots
\]

quare si biaus serici summa aliunde pateat, constans \( C \) definiri poterit. Series autem haece in sequentem geminam rem solvi potest

\[
2nC = \frac{\pi}{k \sin \frac{n\pi}{k}} = \frac{1}{k-n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \ldots
\]

\[
= \frac{1}{k-n} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} + \ldots
\]

§ 195. Cum igitur in \textit{Introductione in Analysein Influi-

torum} pag. 142. ad hanc pervenissem seriem

\[
\frac{\pi}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \ldots
\]

\[
= \frac{1}{2kn \sin \frac{n\pi}{k}} - \frac{1}{2nn}
\]

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(hic scilicet loco litterarum ibi adhibitarum. m. et n. scripsit n. et k).

hoc valore, adhibito, nostra aequatio. erit.

\[ C = \frac{\pi}{2n k \sin \frac{\pi m}{k}} = \frac{\pi}{2n n.} = \frac{\pi}{2n k \sin \frac{\pi x}{k}}. \]

unde fit \( C = \frac{x}{2n n}. \) Hinc ergo, habemus, istam summationem.

\[ \frac{\pi \cos \frac{\eta}{k}}{2n k \sin \frac{\pi}{k}} = \frac{4}{2n n.} = \frac{\cos \gamma}{k \sin \gamma} = \frac{\cos 2\gamma}{4k k \sin n} + \text{et} \text{c.} \]

quae series, utique, omnia attentione digna videtur.

---


§ 196. Quemadmodum in scribus recurrentibus: quilibet terminus ex uno pluribusve praecedentibus secundum legem quandam constantem determinatur; ita ejusmodi series sum consideraturus; in quibus quilibet terminus ex uno pluribusve praecedentibus secundum quamdam legem variabili determinatur. Quoniam autem in talibus scribus formulae generalis singulos terminos exprimens plerumque non est algebraica, sed transcendens singulos terminos per formulas integrales exhiberi conveniet, quae ut valores determinatos praebent, post integracionem quantitati variabili valorem determinatum tribui assumo, ita ut singuli termini prodeant quantitates determinatae; atque—
nunc quaeque principalis hic reedit, quemadmodum istae formulæ integrales debant esse comparatae, ut quilibet terminus secundum datam legem ex uno pluribusve praecedentibus determinetur.

§. 197. Quod quo clarius perspiciatur, contemplemurm seriem notissimam harum formulæ integralium

\[ \int \frac{\partial x}{\sqrt{1-x^2}}, \int \frac{x \cdot x \cdot \partial x}{\sqrt{1-x^2}}, \int \frac{x^4 \cdot \partial x}{\sqrt{1-x^2}}, \int \frac{x^5 \cdot \partial x}{\sqrt{1-x^2}}, \text{ etc.} \]

quaesae singulae ita integrantur, ut evanescant posito \( x = 0 \), tum vero variabili \( x \) tribuatwalor \( = 1 \), quilibet terminus a praecepende ita pendet, ut sit

\[ \int \frac{x \cdot \partial x}{\sqrt{1-x^2}} = \frac{1}{2} \int \frac{\partial x}{\sqrt{1-x^2}}, \]

\[ \int \frac{x^4 \cdot \partial x}{\sqrt{1-x^2}} = \frac{2}{3} \int \frac{x \cdot \partial x}{\sqrt{1-x^2}}, \]

\[ \int \frac{x^5 \cdot \partial x}{\sqrt{1-x^2}} = \frac{5}{6} \int \frac{x^4 \cdot \partial x}{\sqrt{1-x^2}}, \]

atque in genere

\[ \int \frac{x^n \cdot \partial x}{\sqrt{1-x^2}} = \frac{n-1}{n} \int \frac{x^{n-2} \cdot \partial x}{\sqrt{1-x^2}}. \]

Unde patet, hanc formulam generalem spectri posse tanquam terminum generalem illius seriei, atque quemlibet terminum ex praecepende oriri, si iste multiplicetur per \( \frac{n-1}{n} \).

§. 198. Ad similiudinemigitur hujus casus seriem formularum integralium ita in genere constituantamus,

\[ \int \partial u, \int x \cdot \partial u, \int x^2 \cdot \partial u, \int x^3 \cdot \partial u, \int x^4 \cdot \partial u, \text{ etc.} \]

ita ut terminus indici \( n \) respondens sit \( \int x^{n-1} \cdot \partial u \), quae singulae integralia ita accipi sumamus, ut evanescant posito \( x = 0 \), post integrationem autem quantitati variabili \( x \) tribuamus quempiam valorem constantem, veluti \( x = 1 \), vel aliocum quipiam numero. Quibus positis quaeque hic reedit, qualis pro \( u \) assumi debeat functio ipsius \( x \), ut quilibet terminus per unum, vel duo pluresve.
SUPPLEMENTUM V.

praecedentes, secundum legem quandam datam utcunque variabilem, sive ab indice n pendentem, determinet; ubi quidem imprimit co erit respiciendo, ad quot dimensiones index n in scala relationis proposita ascendat; pleurunque autem non ultra primam dimensionem assurgere erit opus. Hinc igitur sequentia problematica pertractamus.

PROBLEMA I.

§ 199. Invenire functionem v, ut ista relationi inter binos terminos sibi succedentes locum habeat:

$$s x^n \partial v = \frac{an + a}{bn + b} s x^{n-1} \partial v.$$

Servatur igitur hic, ut sit

$$s (an + a) x^{n-1} \partial v = s (bn + b) x^n \partial v,$$

si sollecet post integrationem variabili x certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili x iste valor constans fuerit datus, ponamus in generi, dum x est variabilis, hanc equationem locum habere.

$$s (an + a) x^{n-1} \partial v = s (bn + b) x^n \partial v + V,$$

quantitatem autem V ita esse comparatam, ut evanescat postquam variabil ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, ut evanescant posito x = 0; necesse est: ut etiam: ista quantitas, V, eodem quoque casu evanescat.

§ 200. Quoniam haec aequalitas subsistere debet pro omnibus indicibus n, quos quidem semper ut positivos spectamus, facile intelligitur, quantitatem istam V factorem habere debere x^n;
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quo pacto jam isti conditioni satisfiat, ut posito \( x = 0 \) etiam fiat \( V = 0 \). Quamobrem statuamus \( V = x^n Q \), ubi \( Q \) denotet functionem ipsius \( x \) proposto accommodatam, et quam simul ita comparatam esse desideramus, ut evanescat si ipsi \( x \) certus quidem valor tribuatur.

\[ \frac{204.}{ } \text{Cum igitur esse debeat}\]
\[ (a^n + a) \frac{\partial x^{n-1}}{\partial v} = (\beta n + b) \frac{\partial x^n}{\partial v} + x^n Q, \]
differentietur ista aequatio, ae differentiali per \( x^{n-1} \) diviso perseverietur ad hanc aequationem differentialem
\[ (a n^2 + a) \partial v = (\beta n + b) x \partial v + n Q \partial x + x \partial Q, \]
quae cum subsistere debeat pro omnibus valoribus ipsius \( n \), termini ista littera affecti seorsim se tollere debent, unde nanciscimur has duas aequalitates.

I. \((a - \beta x) \partial v = Q \partial x \) et

II. \((a - bx) \partial v = x \partial Q \)

Ex priori \( \partial v = \frac{Q \partial x}{a - bx} \), ex altera vero \( \partial v = \frac{\alpha \partial x}{a - bx} \), qui duos valores inter se aequati suppedient hanc aequationem \( \frac{\partial Q}{Q} = \frac{\partial x}{a - bx} \), quae aequatio resolvitur in has partes
\[ \frac{\partial Q}{Q} = \frac{\partial x}{a} - \frac{\partial x}{b}, \]
cujus ergo integrale erit
\[ lQ = \frac{a}{a} l(x - \frac{a \beta - b a}{a \beta - b a}) \]
unde deducitur:
\[ Q = C x^{a}. \frac{\beta a - \alpha b}{a \beta - b a}. \]

\[ \frac{202.}{ } \text{Ex hoc valore pro} \ Q \ \text{invento statim patet, eum evanescere casu} \ x = \frac{a}{\beta}, \ \text{si modo fuerit} \ \frac{\beta a - \alpha b}{a \beta - b a} < 0; \ \text{sin autem secus eveniat, non patet quomodo haec quantitas ullo casu.} \]
SUPPLEMENTUM V.

3.2

\[ \partial v = \mathcal{C} \frac{a}{x^2} \partial x \left( \alpha - \beta x \right) \frac{\frac{b a - a \beta}{a^2}}{\frac{1}{a^2}} \]

\[ \int x^{n-1} \partial v = \mathcal{C} \frac{n+1}{a} \int x^{\frac{b a - a \beta}{a^2}} \left( \alpha - \beta x \right) \cdot \frac{1}{a^2} \]

ut vero erit

\[ \mathcal{V} = \mathcal{C} \frac{n+1}{a} \left( \alpha - \beta x \right) \frac{b a - a \beta}{a^2} \]

ibi res imprimit eo redit ut ista quantitas praeter casum \( x = 0 \) insuper alio casu evanescent.

\[ \mathcal{C} \text{ o} \text{ r} \text{ o} \text{l} \text{i} \text{u} \text{m} \text{ a} \text{l} \]

§ 203. Hic duo casus occurrent qui peculiarem evolutionem postulant prior est, quo \( \alpha = 0 \) ut autem indeandum erit ab aequatione \( \mathcal{Q} = -\frac{(a-b \xi) \partial x}{\beta \xi} \), unde integrando elicitur

\[ \mathcal{Q} = \mathcal{C} \frac{x^\alpha}{\beta^\beta} \]

qua formula in nihilum abire nequit nisi fiat \( \frac{\alpha}{\beta} = -\infty \), ideoque \( x = 0 \), sicque non duo haberentur casus quibus fieret \( \mathcal{V} = 0 \), cum tamén duo desiderentur. Interim autem hiinc silet

\[ \partial v = \frac{a}{\alpha - \beta x} \frac{b}{\frac{1}{x^\beta}} \partial x \]
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Corollarium 2.

§ 204. Alter casus peculiarem: integrationem postulans erit quo $\beta = 0$; tum autem erit $\frac{dQ}{Q} = \frac{\partial x(\alpha - \beta x)}{\alpha x}$, unde fit: $lQ = \frac{\alpha}{b} l\alpha - \frac{\alpha}{\beta x}$, ideoque $Q = x^{\alpha} \cdot e^{\frac{a}{\beta x}}$, quae formulâ casu $x = \infty$ evanesceit, si modo fuerit $\alpha$ numerus positivus; sin autem $\frac{\alpha}{\beta}$ fuerit numerus negativus, tum $Q$ evanesceit: casu $x = -\infty$. Porro vero hoc casu fiet:

$$\frac{\partial v}{\alpha - \beta x} = \frac{\partial x}{\alpha \cdot x^{\alpha} \cdot e^{\frac{a}{\beta x}}}$$

Sèhio l'io n.

§ 205. His in genere observatis aliquot casus speciales evolvamus, quibus litteris $\alpha$, $\beta$ et $\alpha$, $b$ certos valores tribuemus, qui ad casus jam satis cognitos pertinat.

Exemplum 1.

§ 206. Quaerantur formulae integrales; ut fiat

$$\int x^n \partial v = (\frac{n+1}{2}) \int x^n \partial v.$$  

Cum igitur hic esse debeat

$$(2n+1) \int x^n \partial v = 2n \int x^n \partial v;$$

erit hic casu $\alpha = 2$ et $\alpha = -1$; tum vero $\beta = 2$ et $\beta = 0$; hinc fit:

$$\frac{dQ}{Q} = \frac{\partial x}{2 x (1-x)} = \frac{\partial x}{2 x} \frac{\partial x}{x (1-x)}.$$  

Unde: integrando

$$lQ = \frac{1}{2} l\alpha + \frac{1}{2} l (1-x);$$

ideoque

$$Q = C \sqrt[2]{\frac{1-x}{x}}, \ \text{ergo} \ V = C x^n \sqrt[2]{\frac{1-x}{x}}.$$
Porre cum hic sit \( \partial v \equiv \frac{Q \partial x}{2(1-x)} \), erit

\[
\partial v = \frac{C \partial x \sqrt{1-x}}{2(1-x)} = \frac{C \partial x}{2\sqrt{(1-x)(1-x)}}
\]

sumto ergo \( C = 2 \) erit \( \partial v = \frac{\partial x}{\sqrt{(1-x)(1-x)}} \); et formula nostra generalis

\[
\int x^{n-1} \partial v = \int \frac{x^{n-1} \partial x}{\sqrt{(1-x)(1-x)}}
\]

unde cum sit \( V = x^n \sqrt{1-x} \), haec quantitas manifesto evanescit sumto \( x = 1 \), ita ut nostra formula, si post integrationem statuatur \( x = 1 \), quae est satisfaciat. Quod si jam ponamus \( x = yy \), ista formula induet hanc formam \( 2 \int \frac{y^{2n-2} \partial y}{\sqrt{1-yy}} \), quae, posito post integrationem \( y = 1 \), praebet hanc relationem

\[
\int \frac{y^{2n} \partial y}{\sqrt{1-yy}} = \frac{2n-1}{2n} \int \frac{y^{2n-2} \partial y}{\sqrt{1-yy}}
\]

quae continet relationes supra § 197. commemortas; hinc enim fit

\[
\int \frac{y^{2n} \partial y}{\sqrt{1-yy}} = \frac{3}{2} \int \frac{\partial y}{\sqrt{1-yy}}
\]

\[
\int \frac{y^{2n} \partial y}{\sqrt{1-yy}} = \frac{3}{2} \int \frac{yy \partial y}{\sqrt{1-yy}^2}
\]

\[
\int \frac{y^{2n} \partial y}{\sqrt{1-yy}} = \frac{3}{2} \int \frac{\partial y}{\sqrt{1-yy}}
\]

e tc.

Exemplum 2.

§ 207. Quaerantur formulae integrales, ut fiat

\[
\int x^n \partial v = \frac{a_n}{x^n-1} \int x^{n-1} \partial v.
\]

Cum igitur hic esse debet

\[
(a_n-1) \int x^{n-1} \partial v = a_n \int x^n \partial v,
\]
erit hoo casu $\alpha = -1$, $\beta = \alpha$ et $b = \varphi$, unde per formulas supra datas colligitur

$$Q = C \frac{1}{x^\alpha (a - ax)^\alpha} = C \frac{x^{\alpha-1}}{(1-x)^{\alpha-1}}$$

quae quantitas manifesto evanescit posito $x = 1$. Tum autem erit

$$\partial v = \frac{x^{\alpha} (1-x)^{\alpha} \partial x}{(1-x)}$$

unde formula nostra generalis erit

$$\int x^{n-1} \partial v = \int x^{n-\frac{1}{\alpha} - 1} (1-x)^{\frac{1}{\alpha} - 1} \partial x = \int x^{n-\frac{1}{\alpha} - 1} \partial x$$

qua concinnior redditur, faciendo $x = y^\alpha$, tum enim ea induet hanc formam

$$\int \frac{y^{n-2} \partial y}{(1-y^\alpha)^{\alpha-1}}$$

ubi iterum post integrationem statui debet $y = 1$. Erit hinc

$$\int \frac{y^{n-2} \partial y}{(1-y^\alpha)^{\alpha-1}} = \frac{\alpha n-1}{\alpha n} \int \frac{y^{n-2} \partial y}{(1-y^\alpha)^{\alpha-1}}$$

atque hinc orientur sequentes casus specialis

$$\int \frac{y^{2\alpha-2} \partial y}{(1-y^\alpha)^{\alpha-1}} = \frac{a-1}{a} \int \frac{y^{\alpha-2} \partial y}{(1-y^\alpha)^{\alpha-1}}$$

et

$$\int \frac{y^{3\alpha-2} \partial y}{(1-y^\alpha)^{\alpha-1}} = \frac{2\alpha-1}{2a} \int \frac{y^{2\alpha-2} \partial y}{(1-y^\alpha)^{\alpha-1}}$$

§. 208. Hinc igitur si sumatur $\alpha = 1$, ut fieri debat

$$\int x^n \partial v = \frac{n-1}{n} \int x^{n-1} \partial v,$$
formula nostra generalis jam in \( y \) expressa erit \( f y^{n-2} \partial y \), cujus
ergo valor est \( \frac{n-1}{n-2} y^{n-3} = \frac{n-1}{n-2} \), unde tota series nostrarum
formularum integralium abibit in hanc
\( \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}, \frac{1}{18}, \) etc.

\( \S. \ 209. \) Sumamus etiam \( \alpha = \frac{1}{2} \), et jam non amplius
opus erit ad \( y \) procedere. Hoc igitur casu erit
\( Q = \frac{(1-x)^2}{x^2} \) et \( \partial v = \frac{(1-x)}{x} \partial x \),
unde formula nostra generalis fit
\( \int x^{n-3} \partial v = \int x^{n-3} (1-x) \partial x \),
cujus ergo valor algebraice expressus erit
\( \frac{x}{n-2} x^{n-2} - \frac{x}{n-1} x^{n-1} = \frac{x}{(n-1)(n-2)} \),
unde series nostrarum formularum evadet
\( \frac{1}{0} \cdot \frac{1}{0}, \frac{1}{1} \cdot \frac{1}{1}, \frac{1}{2} \cdot \frac{1}{2}, \frac{1}{3} \cdot \frac{1}{3}, \frac{1}{4} \cdot \frac{1}{4}, \) etc.

\textbf{Exemplum 3.}

\( \S. \ 210. \) Querantur formulae integrales, ut sit
\( \int x^n \partial v = n \int x^{n-1} \partial v \).
Cum igitur esse debat
\( n \int x^{n-1} \partial v = \int x^n \partial v \), erit
\( \alpha = 1, \ a = 0, b = 1, \beta = 0 \).
Cum igitur sit \( \beta = 0 \), casus Coroll. 2. hic locum habet, indeque erit \( Q = e^{-x} \), ideoque \( V = e^{-x} \cdot x^n \), quae quantitas his
dubus casibus evanescit \( x = 0 \) et \( x = \infty \). Porro vero erit
\( \partial v = e^{-x} \partial x \), hincque formula nostra generalis fit \( \int x^{n-1} \partial x \cdot e^{-x} \), unde ipsi seriis termini ab intius sequenti modo se
habebunt
\[ \int e^{-x} \partial x, \int e^{-x} x \partial x, \int e^{-x} xx \partial x, \int e^{-x} x^3 \partial x \text{ etc.} \]
quibus integratis ita ut evanescant posito \( x = 0 \), tum vero posito \( x = \infty \), orietur sequens series satis simplex
\[
1, 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \text{ etc.}
\]
quae est series hypergeometrica Wallisii, cujus ergo terminus generalis est
\[
\int x^{n-1} e^{-x} \, dx = 1 \cdot 2 \cdot 3 \cdot 4 \ldots \ldots (n-1).
\]

§. 244. Ope ergo hujus termini generalis hanc seriem interpolare licebit. Ita si quaeatur terminus medius inter duos primos, ponit debet \( n = \frac{3}{2} \), ac valor hujus termini erit \( \int e^{-x} \partial x \sqrt{x} \), cujus autem valor nullo modo algebraice exprimti potest. Inventi autem singulari modo hunc ipsum terminum aequari \( \frac{\sqrt{\pi}}{2} \), denotante \( \pi \) peripheriam circuli cujus diameter \( = 1 \), unde hic vicissim cognoscimus esse \( \int e^{-x} \partial x \sqrt{x} = \frac{\sqrt{\pi}}{2} \), posito scilicet post integrationem \( x = \infty \). Terminus autem hunc praecedens, indici \( \frac{3}{2} \) respondens, erit \( = \sqrt{\pi} \), cui ergo aequatur formula \( \int \frac{e^{-x} \partial x}{\sqrt{x}} \).

Quod si hic ponamus \( e^x = y \), ita ut posito \( x = 0 \) sit \( y = 1 \), at posito \( x = \infty \) fiat \( y = \infty \), tum ergo ista formula \( \int \frac{e^{-x} \partial x}{\sqrt{x}} \) abit in hanc \( \int \frac{dy}{y^2 \sqrt{y}} \), quae formula si ita integretur ut evanescat posito \( y = 1 \), tum vero fiat \( y = \infty \), praebet valorem ipsius \( \sqrt{\pi} \). Si porro fiat \( y = \frac{1}{2} \), erunt termini integrationis \( z = 1 \), et \( z = 0 \), et formula integralis erit
\[
- \int \frac{dz}{\sqrt{1 - l^2}} \left[ \text{a } z = 1 \right] \left[ \text{ad } z = 0 \right] = \sqrt{\pi},
\]
sive permutatis terminis integrationis erit
\[
\int \frac{dz}{\sqrt{1 - l^2}} \left[ \text{a } z = 0 \right] \left[ \text{ad } z = 1 \right] = \sqrt{\pi},
\]
quemadmodum jam olim observavi.
SUPPLEMENTUM V.

Exemplum 4.

§. 242. Quaerantur formulae integrales, ut sit
\[ \int x^n \, dv = \frac{1}{n} \int x^{n-1} \, dv, \text{ sive} \]
\[ \int e^{x^n} \, dv = n \int e^x \, dv. \]

Hic est \( a = 0 \) et \( a = 1 \), \( \beta = 1 \) et \( b = 0 \); qui ergo est casus in Coroll. 1. tractatus, unde colligitur fore \( Q = e^x \), ideoque \( V = x^n e^{\frac{1}{x}} \), quae formula nequidem evanesce, sumto \( x = 0 \), quandoquidem formula \( e^\frac{1}{x} \) aequivalat infinito infinitismiae potestatis. Hic autem miro modo evenit, ut casus \( x = 0 \) reddat formulam \( e^{-\frac{1}{x}} \) subito evanescementem. Scilicet, si \( \omega \) denotet quantitatem infinit parvam, erit \( e^{\frac{1}{\omega}} = \infty \), tum vero repente fiat \( e^{-\frac{1}{\infty}} = \frac{1}{\infty} = 0 \), quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem \( \partial v = -\frac{e^{x} \partial x}{x} \), ita ut formula nostra generalis futura sit \( -\int x^{n-2} \partial x \, e^{\frac{1}{x}} \), quae autem nobis nullum usum praestare potest.

§. 243. Quod si hic ponamus \( x = y \), formula ista generalis transit in \( \int e^{x} \partial y \). At vero nunc erit \( V = \frac{e^{y}}{y^n} \), quae formula evanescit posito \( y = \infty \). Quomodo cunque autem hanc expressionem transformemus, semper idem incommode occurret. Interim tamen etiam hunc casum sequenti modo resolvere licebit. Sit enim seriei, quam quacrimus, primus terminus \( = \omega \),
ex quò per regulam præscriptam sequentes ordine ita procedent

\begin{align*}
1 & 2 & 3 & 4 & 5 & n \\
\omega & \frac{\omega}{4} & \frac{\omega}{1+2^2} & \frac{\omega}{1+2^3} & \frac{\omega}{1+2^3+4^2} & \cdots & \frac{\omega}{1+2^3+\cdots+(n-1)^{n-1}}
\end{align*}

Supra autem vidimus, hujus formulæ \(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n-1)\) valorem exprimi per hoc integrale \(\int x^{n-1} e^{-x} \, dx\), integratione ab \(x=0\) ad \(x=\infty\) extensa; tantum igitur opus est ut hanc formulam integrale in denominatorem transferamus, et seriei quam quaerimus terminus generalis erit

\[ t \int x^{n-1} e^{-x} \, dx \]

unde satis intelligitur, negotium non per simplicem formulam integrale expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantas \(V\) non duobus casibus evanescere potest; tum enim tantum opus est fractionem \(\frac{a^{n-1}+a}{b^{n-1}+b}\) invertere, atque formulam integrale in denominatorem transferre.

\[ \text{Scholion.} \]

\(\S\) 214. Nisi sit vel \(\alpha = 0\) vel \(\beta = 0\), quos casus, jam expedivimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae \(\alpha\) et \(\beta\) sunt aequales unitati. Cum enim esse debeat

\[ \int x^a \, dv = \frac{a}{b} \int x^{n-1} \, dv, \]

ponatur \(\frac{a}{b} = \gamma\), atque

\[ \frac{a}{b} \int y^n \, dv = \frac{a^{n-1}+a}{b^{n-1}+b} \int y^{n-1} \, dv, \]

quae aequatio reducitur ad hanc formam

\[ \int y^n \, dv = \frac{\frac{n+a}{n+b}}{\beta} \int y^{n-1} \, dv. \]
SUPPLEMENTUM V.

Quod si jam nunc loco \( \frac{a}{\alpha} \) scribamus \( \alpha \), et \( b \) loco \( \frac{b}{\beta} \), resolvenda erit haec formula

\[
\int y^n \, dv = \frac{n+a}{n+b} \int y^{n-1} \, dv,
\]

cujus resolutio, si loco \( x \) scribamus \( y \) et loco litterarum \( \alpha \) et \( \beta \) unitatem, ex superiori solutione praebet primo

\[ Q = C y^\alpha (1-y)^{\beta-a}, \]

quod ergo evanescit posito \( y = 1 \), si modo fuerit \( b > \alpha \), tum autem erit ipsa formula

\[
\int y^{n-1} \, dv = C \int y^{n'+a-1} \, dy \quad (1-y)^{b-a-1};
\]

sin autem fuerit \( b < \alpha \), haec solutio, uti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma \( \int y^{n-1} \, dv \), ita ut tum-esse debeat

\[
\frac{1}{\int y^{n-1} \, dv} = \frac{n+a}{n+b} \cdot \frac{1}{\int y^{n-1} \, dv},
\]

sive

\[
\int y^n \, dv = \frac{n+b}{n+a} \int y^{n-1} \, dv,
\]

cujus resolutio permutatis litteris \( a \) et \( b \) praebet

\[ Q = C y^\beta (1-y)^{\alpha-b}, \]

quae jam casu \( y = 1 \) evanescit, si fuerit \( a > b \), atque tum erit formula generalis

\[
\int y^{n-1} \, dv = C \int y^{n+b-1} \, dy \quad (1-y)^{n-b-1}.
\]

Sive igitur sit \( b > a \) sive \( a > b \), solutio nulla amplius laborat difficultate.

§ 215. Sin autem fuerit vel \( \alpha = 0 \) vel \( \beta = 0 \), loco alterius etiam scribi poterit unitas; unde si esse debeat

\[
\int x^n \, dv = \frac{n+a}{b} \int x^{n-1} \, dv,
\]
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ob $a = 1$ et $\beta = 0$, solutio nostra generalis dat
$$\frac{\partial Q}{Q} = \frac{2x}{x}(a - b\alpha);$$
unde colligitur $Q = Cx^a \cdot e^{b\alpha}$, quae formula evanescit posito $x = \infty$, si modo $b$ fuerit numerus positivus; tum autem fit terminus generalis
$$\int x^{n-1} \partial v = C \int x^{n-1} \partial x \cdot e^{-b\alpha}.$$
At vero numerus $b$ negativus esse nequit, quia aliqùm conditio praescripta esse incongrua.

§ 216. Consideremus etiam alterum casum, quo $a = 0$ et $\beta = 1$, ideoque conditio praescripta
$$\int x^n \partial v = \frac{a}{n+1} \int x^{n-1} \partial v,$$
unde sit
$$\frac{\partial Q}{Q} = -\frac{2x}{x}(a - b\alpha).$$
Hinc autem pro $Q$ orietur valor, qui praeter casum $x = 0$ evanescere non posset; quam ob causam formula generalis statui debet
$$\frac{1}{\int x^{n-1} \partial v};$$
it ut esse debeat
$$\int x^n \partial v = \frac{n+1}{a} \int x^{n-1} \partial v,$$
unde prodict
$$\frac{\partial Q}{Q} = \frac{2x}{x}(b - a\alpha),$$
ideoque $Q = Ce^{-ax} \cdot x^b$, quae expressio evanescit posito $x = \infty$, quoniam $a$ necessario debet esse numerus positivus; tum autem erit
$$\partial v = C e^{-ax} \cdot x^b \partial x,$$
unde formula generalis scribi erit
$$\frac{1}{C \int x^{a+b-1} \partial x \cdot e^{-a\alpha}}.$$
SUPPLEMENTUM V.

Problem 2.

Denotet \( T \) terminum indici \( n \) respondentem \( in \) serie quam considerandum suscepturum, \( at \) vero \( T' \) terminum sequentem, atque proponatur haec conditio adimplenda
\[
T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')} T.
\]

Solutio.

\( \S \) 217. Quoniam hic valores geminati occurrunt, huic conditioni commodissime satisfiet, si terminus generalis \( T \) tanquam productum \( ex \) duobus factoribus spectetur. Statuatur igitur \( T = RS \), sitque terminus sequens \( = R'S' \) et quaeratur formulae \( R \) et \( S \), ut fiat
\[
R' = \frac{\alpha n + a}{\beta n + b} R \quad \text{et} \quad S' = \frac{\alpha' n + a'}{\beta' n + b'} S,
\]
tum enim sumendo \( T = RS \) conditioni praescriptae manifesto satisfiet. Hoc igitur modo pro \( R \) et \( S \) vel hujusmodi formulae
\[
\int x^{n-1} \, dx, \quad \text{vel inversae } \int x^{n-1} \, \frac{1}{x} \, dx
\]
reperientur, id quod pro solutione generali sufficit, unde rem exemplo illustramus.

Exemplum.

\( \S \) 218. Quaeratur formula generalis \( T \), ut fiat
Resolvamus igitur \( T \) in duos factores \( R \) et \( S \), ac statuamus
\[
T' = \frac{n^2 - c}{n^2} T.
\]
\[
R' = \frac{n - c}{n} R \quad \text{et} \quad S' = \frac{n + c}{n} S.
\]
Pro priore forma si statuamus \( R = \int x^{n-1} \, dx \), ex solutione generali, ubi erit \( \alpha = 1, \alpha = -c, \beta = 1 \) et \( b = 0 \), fiet
\[
Q = Cx^{-c}(1-x)^c,
\]
quaes forma manifesto evanesce posito \( x = 1 \), hincque quia fit
\[
V = Cx^{n-c}(1-x)^c,
\]
haec forma etiam casu \( x = 0 \) evanescit, si modo \( n \) fuerit \( > e \); id quod tuto assumi potest, quia exponentem \( n \) successice in infinitum cresceret assumimusi, ac plerumque pro \( c \) fractiones tantum accipi solent. Hinc ergo erit

\[
R = C \int x^n e^{-1} \(1-x\)^{c-1} dx.
\]

\[\text{§ 249. Hinc jam alter valor literae } S \text{ deduci posset, scribeo tantum } \frac{1}{c} \text{ loco } c, \text{ tum autem non amplius fieret } Q = 0 \text{ posito } x = 1, \text{ quamobrem pro } S \text{ formulam inversam}
\]

\[
\frac{1}{\int x^n-1 \partial n} \partial v = \frac{n}{n+c} \int x^n e^{-1} \partial v,
\]

ubi cum sit \( a = 1, \beta = 0, b = c \), reperitur \( Q = C \), quae forma manifesto fit \( = 0 \) posito \( x = 1 \), hinc autem prodict

\[
\partial v = C(1-x)^{c-1} \partial x,
\]
ergo habeamus

\[
S = \frac{1}{C \int x^n e^{-1} \(1-x\)^{c-1} \partial x},
\]

consequentem formulam nostra generalis quaesita erit

\[
T = \frac{\int x^n e^{-1} \(1-x\)^{c-1} \partial x}{\int x^n e^{-1} \(1-x\)^{c-1} \partial x}.
\]

\[\text{§ 229. Quod si ergo nostrae serici per factores procedentes primum terminum ponamus } = A, \text{ ipsa series erit}
\]

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\[
A; \frac{1-cc}{4} A, \frac{1-cc}{4}, \frac{1-cc}{4} A, \frac{1-cc}{4} ; \frac{1-cc}{4} A, \frac{1-cc}{4} ; \frac{1-cc}{4} \cdot \frac{9-cc}{9} A, \text{ etc.}
\]

unde si sumamus \( c = \frac{1}{2} \), crit haec series

\[
A, \frac{1-3}{2} A, \frac{1-3}{2} ; \frac{3-5}{4} A, \frac{1-3}{2} ; \frac{3-5}{4} \cdot \frac{5-7}{6} A, \text{ etc.}
\]

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cujus ergo terminus indici \( n \) respondent est
\[
\int x^{n-\frac{3}{2}} (1-x)^{-\frac{1}{2}} \, dx,
\]
\[
\int x^{n-4} (1-x)^{-\frac{1}{2}} \, dx.
\]
qui posito \( x = yy \) transit in hanc formam
\[
\int y^{2n-2} (1-yy)^{-\frac{1}{2}} \, dy;
\]
\[
\int y^{2n-4} (1-yy)^{-\frac{1}{2}} \, dy.
\]
unde patet, terminum primum fore.
\[
A = \int \frac{\partial y}{\sqrt{(1-yy)}} ; \int \frac{y \partial y}{\sqrt{(1-yy)}} = \frac{\pi}{2} ;
\]
posito scilicet post integrationem \( y = 1 \).

Problem 3.

Denumt \( T \) terminum seriei indici \( n \) respondentem, sintque \( T' \) et \( T'' \) termini sequentes pro indicibus \( n+1 \) et \( n+2 \), si proponatur inter ternos terminos se insequentibus talis ratio, ut sit
\[
(\alpha n + a) T = (\beta n + b) T' + (\gamma n + c) T'',
\]
investigare formulam pro \( T \), qua terminus generalis hujus seriei exprimatur.

Solution.

§ 221. Assumatur pro \( T \) formula integralis \( \int x^{n-1} \, dv \), hujiusque integrale ita capiatur, ut evanescat posito \( x = 0 \), eruntque termini sequentes
\[
T' = \int x^n \, dv \quad \text{et} \quad T'' = \int x^{n+1} \, dv,
\]
si quidem post integrationem variabili \( x \) certus valor determinatus tribuat. Quamdiu autem haec quantitas \( x \) ut variabilis spectatur, ponamus esse
\[
(\alpha n + a) T = (\beta n + b) T' + (\gamma n + c) T'' + x^n Q,
\]
ae perspicuum est \( Q \) ejusmodi functionem esse debeere ipsius \( x \), quae evanescat, si loco \( x \) valor illi determinatus substituatur, quem autem a cifra diversum esse oportet, quoniam jam assumimus, omnes istas formulas in nihilum abire posito \( x = 0 \). Quodsi vero, absuluto calculo, huic conditio nihil modo satisfieri poterit, id erit indicio, problema nostrum hac ratione resolvi non possit, ut salient ejus terminus generalis \( T \) per talem formulam differentialiorem simplicem \( \int x^{n-1} \partial v \) exhibatur.

§ 222. Differentiāmus nunc aequationem modo stabilitam, ac divisione facta per \( x^n \) sequens prohibet aequatio

\[
(\alpha n + a) \partial v = (\beta n + b) x \partial v + (\gamma n + c) xx \partial v + n Q \partial x + x \partial Q,
\]

quaes, quae termini litterae \( n \) affecti seorsum se destruire debent, discernetur in binas sequentes aequationes

\[1^o. \quad \alpha \partial v = \beta x \partial v + \gamma xx \partial v + Q \partial x,\]
\[2^o. \quad \alpha \partial v = \beta x \partial v + \gamma xx \partial v + \partial Q,\]

ex quorum priēre fit

\[\partial v = \frac{Q \partial x}{\alpha - \beta x - \gamma xx},\]

ex altera vero fit

\[\partial v = \frac{\partial Q}{\alpha - \beta x - \gamma xx},\]

quorum valorum posterior per priorem divisus praebet

\[\frac{\partial Q}{Q} = \frac{\partial x (\alpha - \beta x - \gamma xx)}{\alpha - \beta x - \gamma xx},\]

ex cujus ergo integratioe valor ipsius \( Q \) elicere debet; quod facto facile patebit, utrum is certo quodam casu praeter \( x = 0 \) evanescere pessit. Imprimis autem hic notari convenit, si hoc integrale involvat hujusmodi factorem \( e^x \), tum solutionem quoque successu esse caritāram, quandoquidem posito \( x = 0 \) iste factor tantum involvet infiniti potestatem, ut, etiamsi per \( x^n \) multiplicetur, productum etiamnum finitum mancat.
§. 223. Quodsi igitur his conditionibus praeceptis satis-
facere licuerit, tum invento valorem litterae \( Q \), quem ponamus \( x = 0 \), posito \( x \equiv f' \), habebitur:

\[ \partial u = \frac{Q \partial x}{x - \frac{\alpha}{\beta} x - \frac{\gamma}{\alpha} x} \]

et formula generalis natuam serici, completens existit:

\[ T = \int x^{n-1} \partial u = \int \frac{x^{n-1} Q \partial x}{x - \frac{\alpha}{\beta} x - \frac{\gamma}{\alpha} x} \]

quippe cujus integrale, a termino \( x = 0 \) usque ad terminum \( x = f \) extensum, praebebit valorem termini \( T \), indici omnunque, respondit.

§. 224. Inventa autem tali relatione inter ternos termi-
minos cujusiam serici sibi invicem succedentes, inde more solito formari poterit fractio continua, cujus valorem assignare licabit. Si enim characteres \( T', T'', T''', \ldots \), etc: denotent ordine omnes terminos post \( T \) sequentes in infinitum, ex relationibus, quas inter sequentes sequentes formulae deducentur. Ex relatione:

\[ (an+a) \frac{T''}{T'} = (bn+b) \frac{T'}{T''} + (yn+c) \frac{T'}{T''} \]

deducitur

\[ (an+a) \frac{T'}{T''} = \beta n + b + \frac{(\gamma n + c)(an+a)}{(\alpha n + a + \alpha)} \]

Ex. relatione sequente:

\[ (an+a+a) \frac{T'}{T''} = (bn+b+b) \frac{T'}{T''} + (yn+c) \frac{T'}{T''} \]

deducitur

\[ (an+2a+a) \frac{T''}{T'} = \beta n + \beta + b + \frac{(\gamma n+2\gamma+c)(an+2\alpha+a)}{(\alpha n + 3\alpha + a)} \frac{T'}{T''} \]

Simili modo sequentes relationes suppeditabunt

\[ (an+2a+a) \frac{T'}{T''} = \beta n + 2\beta + b + \frac{(\gamma n+2\gamma+c)(an+3\alpha+a)}{(an+3\alpha+a)} \frac{T'}{T''} \]
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\[(an+3x+a)^T = (bn+3b+2\beta + b)^T + \frac{(\gamma n+3\gamma+c)(xn+3x+a)}{(an+4x+a)T^m} T^{mn}, \text{ etc.}\]

unde manifestum est, si in prima formula continuo sequentes valores ordine substituantur, prodituram esse fractionem continuum, cujus valor acqualis erit formulae \[(an+3x+a)^T.\]

§ 225. Quod si ergo loco \(n\) successive scribamus numeros \(1, 2, 3, 4, \text{ etc.}\) sequens problema circa fractiones continuas resolvere poterimus.

\[\text{Proposita fractione continua hujus formae:} \]
\[
\begin{align*}
\beta + b + (\gamma + c)(2x+a) \\
2\beta + b + (2\gamma + c)(3x+a) \\
3\beta + b + (3\gamma + c)(4x+a) \\
4\beta + b + (4\gamma + c)(5x+a) \\
5\beta + b + (5\gamma + c)(6x+a) \\
6\beta + b + \text{ etc.}
\end{align*}
\]

\[\text{ejus valorem investigaret.}\]

§ 226: Consideretur in genere ista relation inter ternas quantitates sibi successentes \(T, T', T''\), quae sit
\[(an+3x+a):T = (bn+b):T' + (\gamma n+c):T'', \]
atque ex praecedente problemate quaeratur valor ipsius \(T\), siquisdem fieri potest, hoc modo expressus:

\[T = \int x^{n-1} \vartheta \, dx = \int \frac{x^{n-1}}{\alpha - \beta x - \gamma x^2} \, dx,\]

cujus integrale ab \(x = 0\) usque ad \(x = f\) extendatur, qua formula inventa ponatur.
SUPPLEMENTUM V.

\[ \int \frac{Q \, dx}{a - \beta x - \gamma x^2} = A \quad \text{et} \quad \int \frac{Q \, dx}{a - \beta x - \gamma x^2} = B, \]

ita ut \( A \) et \( B \) sint valvae ipsius \( T \), pro casibus \( n = 1 \) et \( n = 2 \), quibus definitis fractionis continuae propositae valor per praeecedentia erit \( \frac{(a + x)A}{B} \). Hanc igitur investigationem ad sequentia exempla accommodemus.

**Exemplum 4.**

§. 227. Investigem valorem fractionis continuae notissimae, quam olim \textit{Brounerus} pro quadratura circuli profulit, quae est

\[
\frac{2 + 1 \cdot \frac{d}{d}}{2 + \frac{3 \cdot 3}{2 + 5 \cdot 5}} \quad \frac{2 + \text{etc.}}{2 + \text{etc.}}
\]

Quia omnes partes integrae laevam respicientes sunt constantes \( = 2 \), pro nostra forma generali fit

\( \beta + b = 2, \ 2\beta + b = 2, \ 3\beta + b = 2 \), etc.

ergo \( \beta = 0 \) et \( b = 2 \); et pro numeratibus sequentium fractionum, quandoquidem constant binis factoribus, erit pro factoribus prioribus

\( \gamma + c = 4, \ 2\gamma + c = 3, \ 3\gamma + c = 5 \), \( 4\gamma + c = 7 \), etc.

unde concluditur \( \gamma = 2 \) et \( c = -1 \), pro alteris vero erit

\( 2\alpha + a = 1, \ 3\alpha + a = 3, \ 4\alpha + a = 5 \), etc.

unde \( \alpha = 2 \) et \( a = -3 \). Ex his autem valoribus colligimus hanc equationem

\[
\frac{\partial Q}{Q} = -\frac{3x(3+2x-x^2)}{2x(1-x^2)},
\]

quae per \( 1-x \) depressa praebet

\[
\frac{\partial Q}{Q} = -\frac{3x(3-x)}{2x(1-x)},
\]
unde integrando fit

\[ lQ = - \frac{3}{2} l x + l (1 - x) \quad \text{et hinc} \quad Q = \frac{4 - x}{x^2}, \]

ex quo valore porro sequitur

\[ A = \int \frac{(1 - x) \partial x}{2 x^3 (1 - x x)} = \int \frac{\partial x}{2 x (1 + x) \sqrt{x}} \]

\[ B = \int \frac{(1 - x) \partial x}{2 x^3 (1 - x x)} = \int \frac{\partial x}{2 (1 + x) \sqrt{x}}. \]

§. 228. In his autem valoribus istud incommodum deprehenditur, quod prius integrale evanescentis reddi nequit posito \( x = 0 \). Hoc autem incommodum facile removeri potest, si fractionem continuam supremo membro truncussem et quæramus valorem istius fractionis

\[ 2 + \frac{3}{2} + \frac{5}{5} + \frac{7}{5} + \text{etc.} \]

qui si repertus fuerit \( s \), erit ipsius propositae valor \( b + \frac{4}{5} \)

Nunc vero, comparatione instituta, fit quidem ut ante \( \beta = 0 \) et \( b = 2 \), tum vero \( y = 2 \) et \( c = -1 \), \( a = 2 \) et \( a = -1 \), unde sequitur

\[ \frac{\partial Q}{\partial x} = \frac{\partial (1 + 2 x + x x)}{x (1 - x x)} = \frac{\partial x (1 + x)}{2 x (1 - x)}, \]

unde integrando fit

\[ lQ = - \frac{4}{3} l x + l (1 - x), \quad \text{ideoque} \quad Q = \frac{4 - x}{y^2}; \]

ex quo valore jam habebimus

\[ A = \int \frac{(1 - x) \partial x}{2 (1 - x x) \sqrt{x}} = \frac{1}{2} \int \frac{\partial x}{(1 + x) \sqrt{x}} \quad \text{et} \]

\[ B = \frac{1}{2} \int \frac{\partial x \sqrt{x}}{1 + x}. \]
SUPPLEMENTUM V.

ubi cum sit $Q = \frac{1 - x}{\sqrt{x}}$, ejus valor manifeste evanescit poste $x \rightarrow a$, quomobrem illa integralia a termino $x = 0$ usque ad $x = a$ sunt extendenda.

§. 229. Quo nunc haec integralia facilius eramus, statuamus $x = \pi z$, ita ut termini integrationis etiam nunc sint $z = 0$ et $z = 1$, est

$$ A = \int \frac{dz}{1 + z^2} = \arctan z = \frac{\pi}{2}, \text{ et} $$

$$ B = \int \frac{z^2 dz}{1 + z^2} = 1 - \frac{\pi}{2}, $$
sicque habebimus $s = \frac{\pi}{4}$, quocirca ipsius fractionis Brownecherianae valor est $1 + \frac{\pi}{4}$, omnino ut olim Brownerheus jam venerat.

Exemplum 2.

§. 230. Investigare valorem hujus fractionis continuae Brownecherianae latius patentis

$$ b \rightarrow 1.1 $$

$$ \frac{b \rightarrow 3.3}{b \rightarrow 5.5} $$

$$ \frac{b \rightarrow \text{etc.}}{b \rightarrow \text{etc.}} $$

Ut hic incommodum superius evitemus, omittamus membro supramum et quacramus

$$ s = b + 3.3 $$

$$ \frac{b \rightarrow 5.5}{b \rightarrow \text{etc.}} $$

quandoquidem tum exit valor quaesitas $= b + \frac{\pi}{4}$. Nunc igitur exit $\beta = 0$ et $b = b$, $\gamma = 2$, $c = 1$, $a = 2$ et $a = -1$, unde sit
\[ \frac{\partial Q}{\partial \xi} = -\frac{\partial (1+\xi^2)}{2x(1-\xi^2)}, \]

ac proinde

\[ Q = -\frac{\xi}{4} + \frac{\xi - \frac{b-2}{4}}{L(1+x) + \frac{b+2}{4} L(1-x)}, \]

hincque

\[ Q = \frac{\xi}{4} \frac{b+2}{(1-x)^{\frac{b-2}{4}}} \cdot \frac{1}{\sqrt{x}} \]

quae formula manifesto fit \( 1 \) ponendo \( x = 1 \), siquidem \( b+2 \)

fuerit numeros positivos, unde fit

\[ \frac{\partial v}{\partial \xi} = \frac{(1-x)^{\frac{b-2}{4}}}{2(1+x)^{\frac{b+2}{4}}} \frac{\partial x}{\sqrt{x}} \]

Hinc autem colligetur

\[ A = \frac{1}{2} \int (1-x)^{\frac{b-2}{4}} \frac{\partial x}{\sqrt{x}} \]

et

\[ B = \frac{1}{2} \int (1-x)^{\frac{b-2}{4}} \frac{\partial x}{\sqrt{x}} \]

sive ponendo \( x = zz \) habeamus

\[ A = \int (1-zz)^{\frac{b-2}{4}} \frac{\partial z}{\sqrt{zz}} \]

et

\[ B = \int (1-zz)^{\frac{b-2}{4}} \frac{\partial zz}{\sqrt{zz}} \]

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quae ambo integralia a z = 0 usque ad z = 1 sunt extendenda. Ex his autem valoribus A et B erit e = \frac{A}{B}; ipsius igitur fractionis propositae valor erit e = b + \frac{1}{2} = b + \frac{B}{A}.

§. 231. Quod si hie ponatur b = 2, probit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes \frac{b-2}{4} et \frac{b+2}{4} non sunt numeri integri, tum litteras A et B neque per arcus circulares, neque per logarithmos exprimere licet. Veluti si fuerit b = 4, erit

\[ A = \int \frac{\partial z \sqrt{(1 - 2z^2)}}{(1 - 2z^2)^{3/2}} \] 

cujus valor per arcus ellipticos exhiberi possit. At si b fuerit numerus impar, hi valores multo magis evadunt transcendentes, ita ut his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi sint numeri integri, totum negotium per arcus circulares expedire licebit.

§. 232. Exponentes autem illi \frac{b-2}{4} et \frac{b+2}{4} erunt numeri integri, quoties fuerit b numerus hujus formae b = 4i + 2, tum enim erit

\[ A = \int \frac{(1 - z^2)^i}{(1 + z^2)^{i+1}} \partial z \] 
\[ B = \int \frac{(1 - z^2)^i z z \partial z}{(1 + z^2)^{i+1}}; \] 

quos ergo casus quomodo evolvi oporteat, operae pretium erit docere, quoniam Wallisius eos jam est contemplatus.
§. 233. Quoniam hoc negotium toium redit ad reductionem hujusmodi formulæ integralium ad formæ simpliciores, consideremus in genere formam

\[ P = \frac{z^n}{(1 + zz)^n} \]

cujus differentiale sub sequentibus formis exhiberi potest.

1) \[ \partial P = \frac{m z^{m-1} \partial z}{(1 + zz)^{n+1}} - \frac{2 n z^m \partial z}{(1 + zz)^{n+2}} \]

2) \[ \partial P = \frac{m z^{m-1} \partial z}{(1 + zz)^{n+1}} - \frac{(2n - m) z^m \partial z}{(1 + zz)^{n+2}} \]

3) \[ \partial P = \frac{(2n - m) z^{m-1} \partial z}{(1 + zz)^2} - \frac{2 n z^m \partial z}{(1 + zz)^{n+1}} \]

unde hanc triplex reducimus integralum deducimus.

I. \[ \int \frac{z^{m-1} \partial z}{(1 + zz)^{n+1}} = \frac{m}{2n} \int \frac{z^{m-1} \partial z}{(1 + zz)^n} - \frac{1}{2n} \frac{z^m}{(1 + zz)^n} \]

II. \[ \int \frac{z^{m+1} \partial z}{(1 + zz)^{n+1}} = \frac{m}{2n-m} \int \frac{z^{m-1} \partial z}{(1 + zz)^n} - \frac{1}{2n-m} \frac{z^m}{(1 + zz)^n} \]

III. \[ \int \frac{z^{m-1} \partial z}{(1 + zz)^{n+1}} = \frac{2n-m}{2n} \int \frac{z^{m-1} \partial z}{(1 + zz)^n} + \frac{1}{2n} \frac{z^m}{(1 + zz)^n} \]

quorum reductionem ope casibus \( b = 4i + 2 \) totum negotium absolvit et ad formulam \( \frac{z}{x} \) reduci poterit, siquidem post integrationem sumatur \( x = 1 \).

§. 234. Sit \( i = 1 \) idque \( b = 6 \), eritque

\[ A = \int \frac{(1 - zz) \partial z}{(1 + zz)^2} \]

et \( B = \int \frac{(1 - zz) \partial z}{(1 + zz)^2} \).

Nunc igitur reperiemus per reductionem tertiam

\[ \int \frac{\partial z}{(1 + zz)^2} = \frac{1}{2} \int \frac{\partial z}{1 + zz} + \frac{1}{2} \cdot \frac{z}{1 + zz} = \frac{z}{2} + \frac{1}{2} \]
et per reductionem" primam

\[ \int \frac{zx \, dz}{(1+z^2)^2} = \frac{1}{2} \int \frac{dz}{1+z^2} - \frac{1}{4} \cdot \frac{z}{1+z^2} = \frac{\pi}{4} - \frac{1}{4}, \]

porro

\[ \int \frac{zx \, dz}{(1+z^2)^2} = \frac{3}{2} \int \frac{zx \, dz}{1+z^2} - \frac{1}{4} \cdot \frac{z^3}{1+z^2} = \frac{5}{8} - \frac{3\pi}{8}. \]

Ex his jam valoribus colligitur \( A = \frac{1}{2} \) et \( B = \frac{7}{2} - \frac{3\pi}{2} \), ideoque

\[ \frac{B}{A} = \frac{\pi}{3} - 3, \] quocirca orietur ista summatio

\[ 3 + \frac{\pi}{3} = 6 + 1.1 \]

\[ 6 + \frac{3}{3} = \frac{6 + 5.5}{6 + 7.7} = \frac{6}{6 + \text{etc.}}. \]

\[ \text{§ 235.} \quad \text{Sit nunc} \quad i = 2 \quad \text{et} \quad b = 10, \quad \text{critique} \]

\[ A = \int \frac{(1-zx)^2 \, dx}{(1+z^2)^2} \quad \text{et} \quad B = \int \frac{zx (1-zx)^2 \, dx}{(1+z^2)^3}. \]

Quo harum integralium valores investigemus, sequentes evolvamus formulas

\[ \int \frac{dz}{(1+z^2)^2} = \frac{1}{4} \int \frac{dz}{1+z^2} + \frac{1}{4} \cdot \frac{z}{1+z^2} = \frac{\pi}{32} + \frac{1}{4} \]

\[ \int \frac{z \, dx}{(1+z^2)^2} = \frac{1}{4} \int \frac{z \, dx}{1+z^2} - \frac{1}{4} \cdot \frac{z^2}{1+z^2} = \frac{\pi}{32} \]

\[ \int \frac{z^4 \, dx}{(1+z^2)^2} = \frac{3}{4} \int \frac{z^4 \, dx}{1+z^2} - \frac{1}{4} \cdot \frac{z^5}{1+z^2} = \frac{3\pi}{32} - \frac{5}{4} \]

\[ \int \frac{z^5 \, dx}{(1+z^2)^2} = \frac{5}{4} \int \frac{z^5 \, dx}{1+z^2} - \frac{1}{4} \cdot \frac{z^6}{1+z^2} = \frac{5\pi}{32} - \frac{15}{4}. \]

Ex quibus jam valoribus deducetur \( A = \frac{4}{5} \) et \( B = 2 - \frac{5\pi}{8} \), ideoque

\[ \frac{B}{A} = \frac{46 - 5\pi}{\pi}, \] unde emergit sequens summatio

\[ \frac{5\pi + 16}{\pi} = 10 + 1.4 \]

\[ 10 + \frac{3}{3} = \frac{10 + 5.5}{10 + \text{etc.}}. \]
AD TOM. I. CÀP. VIII.

§. 236. Si \( b \) esset numerus negativus, investigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

\[
s = - \frac{a + \alpha}{b + \beta} - \frac{c + \gamma}{d + \delta} - e + \text{etc.}
\]

semper erit

\[
- s = a + \alpha - \frac{b + \beta}{c + \gamma - \frac{d + \delta}{e + \text{etc.}}}
\]

unde si habeatur valor istius expressionis, idem negative sumtus dabit valorem illius.

Exemplum 3.

§. 237. Proposita sit fractio continua, cujus valorem investigari oporteat, ista.

\[
\begin{align*}
1 + 1.1 & \frac{1}{3 + 3.3} \frac{5 + 5.5}{7 + 7.7} \frac{9 + \text{etc.}}{10 + \text{etc.}}
\end{align*}
\]

Quo fractiones supra allegatae, omissi membro supremo, sint
SUPPLEMENTUM V.

\[ \frac{3 + 3 \cdot 3}{5 + 5 \cdot 5} = \frac{7 + 7 \cdot 7}{9 + \text{etc.}} \]

critque \( \beta + b = 3 \), \( 2\beta + b = 5 \), ideoque \( \beta = 2 \) et \( b = 1 \);
tum vero ut ante \( a = 2 \), \( a = -1 \), \( \gamma = 2 \) et \( e = -1 \); invento autem \( s \) erit valor quaeatur \( = 1 + \frac{3}{5} \). Nunc igitur habebimus

\[ \frac{\partial Q}{Q} = -\frac{3 (1 + x + xx)}{2x (1 - x - xx)} \]

Est vero

\[ \frac{1 + x + xx}{x (1 - x - xx)} = \frac{1}{2} + \frac{2 + 2x}{1 - x - xx}, \]

unde fit

\[ l Q = -\frac{1}{2} lx = \int \frac{3 (1 + x)}{1 - x - xx} \]

Porro vero pro formula \( \int \frac{3 (1 + x)}{1 - x - xx} \) invenienda, statuamus denominatorem

\[ 1 - x - xx = (1 - f x) (1 - g x), \]

critque \( f + g = 1 \) et \( fg = -1 \), unde fit

\[ f = \frac{1 + \sqrt{5}}{2} \] \( \text{et} \) \( g = \frac{1 - \sqrt{5}}{2} \).

Nunc statuatur

\[ \frac{1 + x}{1 - x - xx} = \frac{f}{1 - f x} + \frac{g}{1 - g x}, \]

unde reperietur

\[ \Upsilon = \frac{1 + f}{f - g} \] \( \text{et} \) \( \Theta = \frac{(1 + g)}{f - g} \),

sive substitutus pro \( f \) et \( g \) valoribus supra datis erit

\[ \Upsilon = \frac{\sqrt{5} + 3}{2\sqrt{5}} \] \( \text{et} \) \( \Theta = \frac{\sqrt{5} - 3}{2\sqrt{5}} \),

quibus inventis erit
AD TOM. I. CAP. VIII.

\[ \int \frac{2x}{1-x-x^2} \, dx = \frac{3}{f} \int (1-fx) - \frac{3}{g} \int (1-gx) = \]
\[ - \frac{(1+\frac{\sqrt{5}}{2})}{2\sqrt{5}} \int (1-fx) - \frac{(\sqrt{5}-1)}{2\sqrt{5}} \int (1-gx), \]
quocirca sit
\[ lQ = \frac{1}{6} l\alpha + \frac{(\sqrt{5}+1)}{2\sqrt{5}} l(1-f\alpha) + \frac{(\sqrt{5}-1)}{2\sqrt{5}} l(1-g\alpha), \]
consequentem
\[ Q = \left( \frac{\sqrt{5}+1}{2\sqrt{5}} \right) \frac{\sqrt{5}+1}{2\sqrt{5}} \]
qui valor duobus casibus evanescit: altero quo
\[ \alpha = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}, \]
altero vero quo \( \alpha = \frac{1}{\sqrt{5}} = -\frac{1+\sqrt{5}}{2} \); utrovis autem utamur, res eodem redibit.

§ 238. Ex hoc autem valore habebimus
\[ A = \int \frac{Q\partial x}{1-x^2-x}, \quad B = \int \frac{Q\partial x}{1-x^2-x}, \]
unde porro deducitur
\[ s \equiv (\alpha+a) \frac{A}{B} = \frac{A}{B}, \]
et propositae fractionis summa erit \( \frac{B}{A} + \frac{B}{A} \). Hinc autem nihil ulteriorius concludere licet, ob formulas differentiales non solum irrationalles, sed etiam vere transcendentes ob exponentes surdos.

EXEMPLUM 4.

§ 239. Proposita sit hæc fractio continua
\[ b + \frac{1}{2}, \quad b + \frac{2}{2}, \quad b + \frac{3}{3}, \quad b + \frac{4}{4}, \quad b + \text{etc.} \]
ubi est \( \beta = 0, \ b = b \). Nunc consideremus hanc formam
\[
s = b + 2 \cdot \frac{2}{b + 3} \cdot \frac{3}{b} \quad \text{etc.}
\]
quippe quo valore invento quaestus erit \( b + \frac{1}{3} \). Habeimus
igitur \( \gamma + c = 2, \ 2\gamma + c = 3 \), ideoque \( \gamma = 1 \) et \( c = 1 \),
deinde erit \( a = \gamma = 1, \ a = 0 \) et \( c = 1 \). Hinc igitur colligimus
\[
\frac{\partial Q}{Q} = -\frac{\partial x (bx + xx)}{x (1-xx)} = -\frac{\partial x (b+xx)}{1-xx},
\]
ideoque
\[
LQ = -\frac{1}{5} \ln \frac{1 - x}{1 - x/3} + \ln (1 - xx),
\]
hincque
\[
Q = \frac{(1-x)^2 \sqrt{1 - xx}}{(1 + x)^2} = \frac{(1-x)^{\frac{b+1}{b-1}}}{(1+x)^{\frac{b-1}{b+1}}},
\]
quaer quaequantitas manifesto evanescit positio \( x = 1 \). Hinc igitur fier
\[
A = \int_{1-xx}^{x} Q \partial x = \int_{1}^{\frac{b+1}{b-1}} \frac{(1-x)^{\frac{b+1}{b-1}}}{\frac{b-1}{b+1}} \partial x = \int_{1-x}^{\frac{b+1}{b-1}} \frac{(1-x)^{\frac{2}{b+1}}}{(1+x)^{\frac{2}{b-1}}} \partial x,
\]
\[
B = \int_{\frac{b-1}{b+1}}^{x} \frac{(1-x)^{\frac{2}{b+1}}}{\frac{b-1}{b+1}} \partial x,
\]
tum autem erit \( s = (a + a) \frac{A}{B} = \frac{A}{B} \), ideoque summa quaesita
\[
= b + \frac{B}{A}.
\]
\[\S\ 240.\ \] Percurramus nunc casus praeceptuos: ac primo
sit \( b = 1 \), ideoque
AD TOM. I. CAP. VIII.

\[ A = \int \frac{\partial x}{i+x} = l \left( 1 + x \right) = 12, \text{ et} \]
\[ B = \int \frac{x \partial x}{i+x} = x - \int \frac{\partial x}{i+x} = 1 - l2, \]
ideoque \( b + \frac{B}{A} = \frac{1}{12} \); ergo hinc prodiit ista summatio
\[ \frac{1}{12} = 1 + \frac{1 \cdot 1}{1 + 2 \cdot 2} \]
\[ \frac{1 + 3 \cdot 3}{1 + \frac{4}{3}}, \text{ etc.} \]

§ 241. Sit nunc \( b = 2 \), critque
\[ A = \int \frac{\partial x \sqrt{1-x}}{(1+x^2)^3}, \text{ et } B = \int \frac{x \partial x \sqrt{1-x}}{(1+x^2)^3}. \]

Ad has formulas rationales reddendas statuamus
\[ \sqrt{1-x} \sqrt{1+x} = z, \text{ critque } x = \frac{1-z^2}{1+z^2}, \]
unde terminus integrationis \( x = 0 \) et \( x = 1 \) respondebant \( z = 1 \)
et \( z = 0 \); tum vero erit
\[ 1 + x = \frac{z}{1+z^2}, \text{ et } \partial x = -\frac{4z \partial z}{(1+z^2)^2}, \]
hincque colligitur
\[ A = -2 \int \frac{z \partial z}{1+z^2} = -2z + 2 \operatorname{Arc tang} z + 2 \frac{\pi}{2} = 2 - \frac{\pi}{2}, \]
porro fit
\[ B = -2 \int \frac{z \partial z}{(1+z^2)^2} + 2 \int \frac{4z \partial z}{(1+z^2)^2}. \]

Per reductiones igitur supra § 234. monstratas, si hic scilicet terminos integrationis \( z = 1 \) et \( z = 0 \) permutemus, ut habeamus
\[ B = -2 \int \frac{z \partial z}{(1+z^2)^2} - 2 \int \frac{4z \partial z}{(1+z^2)^2}, \text{ erit} \]
\[ B = 2 \left( \frac{\pi}{8} - \frac{1}{4} \right) - 2 \left( \frac{3}{4} - \frac{\pi}{4} \right) = \pi - 3, \]

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unde sequitur ista summatio.
\[
\frac{\sqrt{2}}{\sqrt{2} + \sqrt{3}} = 2 + \frac{1}{2 + \frac{2}{2 + \frac{3}{3 + \frac{4}{\ldots}}}}
\]
quae Brouncherianae simplicitate nihil cedit.

§ 242. Si ponamus $b = 0$, fractio continua, abit in sequens continuum productum
\[
\frac{1 \cdot 1}{2 \cdot 2} \cdot \frac{3 \cdot 3}{4 \cdot 4} \cdot \frac{5 \cdot 5}{6 \cdot 6} \cdot \frac{7 \cdot 7}{8 \cdot 8} \cdot \ldots
\]
hoc autem casu fit
\[
A = \int \frac{x}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} \quad \text{et} \quad B = \int \frac{x \, dx}{\sqrt{1-x^2}} = 1;
\]
unde istius producti valor colligitur \(\frac{n}{\pi}\), id quod egregie convenit eum jam dudum cognitis, quandoquidem hoc productum est ipsa progressio Wallisiana.

Exemplum 5.

§ 243. Proposita sit haec fractio continua, ubi $b = 0$,
\[
b + 1
\]
\[
b + \frac{3}{b + 6}
\]
\[
b + \frac{10}{b + \ldots}
\]
Omissō supremo membro statuamus.
\[
s = b + 3
\]
\[
b + \frac{5}{b + \ldots}
\]
et primo numeratores per producta representamus hoc modo
\[ 3 = 2 \cdot \frac{3}{2}; \ 6 = 3 \cdot \frac{2}{3}; \ 10 = 4 \cdot \frac{5}{4}, \text{ etc.} \]
quorum priores comparentur cum formula
\[ \gamma + c, \ 2\gamma + c, \ 3\gamma + c, \text{ etc.} \]
posteriores vero cum formula \[ 2a + a, \ 3a + a, \ 4a + a, \text{ etc.} \]
neque \[ \gamma = 1, \ c = 1, \ a = \frac{1}{2}, \text{ a = } \frac{1}{2}, \text{ unde erit} \]
\[ \frac{\partial Q}{Q} = \frac{\partial x}{x \left( \frac{1}{2} - bx - ax \right)} = \frac{\partial x}{x \left( 1 - 2bx - 2ax \right)}, \text{ sive} \]
\[ \frac{\partial Q}{Q} = \frac{\partial x}{x} \frac{2b \partial x}{1 - 2ax}, \]
cujus integrale est
\[ lQ = l\frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} l\frac{1 + ax}{1-x x}, \text{ ergo} \]
\[ Q = \frac{x \left( 1 - x \sqrt{2} \right)^{\sqrt{2}}}{b}, \]
\[ (1 + x \sqrt{2})^{\sqrt{2}} \]
quae formula evanesceit casu \[ x = \frac{\sqrt{2}}{2} \]. Hinc igitur erit
\[ \partial v = \frac{2x \left( 1 - x \sqrt{2} \right)^{\sqrt{2}} \partial x}{b}, \]
\[ (1 - 2ax) \left( 1 + x \sqrt{2} \right)^{\sqrt{2}} \]
Sit \[ \frac{\sqrt{2}}{2} = \lambda, \] neque
\[ A = 2 \int \frac{x \left( 1 - x \sqrt{2} \right)^{\lambda} \partial x}{(1 - 2ax) \left( 1 + x \sqrt{2} \right)^{\sqrt{2}}} = \int \frac{x \left( 1 - x \sqrt{2} \right)^{\lambda-1} \partial x}{(1 + x \sqrt{2})^\lambda}, \]
et
\[ B = 2 \int \frac{x \ln \left( 1 - x \sqrt{2} \right)^{\lambda-1} \partial x}{(1 + x \sqrt{2})^\lambda}, \]
ubi post integrationem statuitur \[ x = \frac{1}{\sqrt{2}} \]; tum autem sit \[ s = \frac{A}{B}, \]
hincque valor fractionis propositae \[ b + \frac{b}{A} \],

\[ 52^* \]
SUPPLEMENTUM V.

§ 244. Nisi igitur fuerit \( \lambda = \frac{b}{\sqrt[2]{2}} \) numerus rationalis, hos valores commode assignare non licet. Sit igitur \( b = \sqrt[2]{2} \), sive \( \lambda = 1 \), et rite

\[ A = 2 \int \frac{x \, dx}{(1 + x \sqrt[2]{2})^2}, \quad \text{et} \quad B = 2 \int \frac{xx \, dx}{(1 + x \sqrt[2]{2})^2}. \]

Hinc integrando colligitur

\[ A = i \left(1 - x \sqrt[2]{2}\right) - \frac{\sqrt[2]{2}x}{1 + x \sqrt[2]{2}}, \]

ideoque posito \( x \sqrt[2]{2} = 1 \) fiet \( A = l2 - \frac{1}{2} \); tum vero reperitur \( B = \frac{\sqrt[2]{2}}{2} - \sqrt[2]{2} \cdot l2 \);

quare ob \( b = \sqrt[2]{2} \) erit

\[ b + \frac{B}{A} = \sqrt[2]{2} (2 \sqrt[2]{2} - 1)^2, \]

unde sequitur haec summatio

\[ \frac{1}{\sqrt[2]{2} (2 \sqrt[2]{2} - 1)} = \sqrt[2]{2} + 1 \]

\[ \frac{3}{3 \sqrt[2]{2} + 6} \]

\[ \sqrt[2]{2} + \text{etc.} \]

Scholion.

§ 245. Fractiones autem continuae, ad quas praejectas calculus numerico deducimur, hujusmodi formam habere solent

\[ a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \text{etc.}}}}}. \]

ubi omnes numeratorem sunt unitates; denominatores vero \( a, b, c, d, e \), etc. numeri integri. Verum igitur nostrae methodi difficulter
talium formarum valores eruere licet, etiamsi numeri α, β, c, d, etc.
progressionem arithmetican constituant, id quod sequenti exemplo
ostendamus.

Exemplum 6.

§. 246. Proposita sit ista fractio continua

\[
\frac{\beta + b + 1}{2\beta + b + 1}
\]

\[
\frac{3\beta + b + 1}{4\beta + b + 1}
\]

ubi α = 0, γ = 0, α = t, c = t.

Hinc fit

\[
\frac{\partial Q}{\partial x} = \frac{\partial x (1 + b x - xx)}{\beta xx}, \text{ unde}
\]

\[
\frac{1}{\beta x} + \frac{b}{\beta} \frac{1}{x} + \frac{x}{\beta} \text{ et}
\]

\[
Q = e^{\frac{1}{\beta xx}} \cdot xx^\beta,
\]

quae autem expressione millo causu evanesce potest, etiamsi per
x^n multiplicetur, siquidem β fuerit numerus positivus. Vetum si
pro β sumamus numeros negativos, puta β = -m, tum valor

\[
Q = x^m \times e^{-m x}, \text{ manifesto evanescit, tam si } x = 0, \text{ quam}
\]

si x = ∞. Hinc autem erit,

\[
\frac{\partial n}{\partial x} = \frac{x^m \cdot e^{m x}}{m xx},
\]

quamobrem habebimus

\[
A = \frac{1}{m} \int_2^{\frac{1 + b}{m}, \text{ etr}} \frac{\partial x}{x}.
\]
SUPPLEMENTUM V.

\[ B = \frac{1}{m} \int \frac{\partial x}{x^{1+\frac{b}{m}} \cdot e^{mx}}. \]

His valoribus inventis formula \( \frac{A}{B} \) exprimet summan hujus fractionis continuae

\[-m+b+1 \quad \frac{-2m+b+1}{-3m+b+1} \quad \frac{-4m+b+1}{-5m+b+1} \quad \text{etc.}\]

quamobrem formula illa negative sumta \(-\frac{A}{B}\) exprimet valorem hujus fractionis continuae

\[ m-b+1 \quad \frac{2m-b+1}{3m-b+1} \quad \frac{4m-b+1}{\text{etc.}} \]

quem igitur assignare liceret, si modo formulæ integrales \( A \) et \( B \) expediri et a termino \( x = 0 \) ad \( x = \infty \) extendi possent. Verum istae formulæ ita sunt comparatae, ut earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impediat, quo minus fractio \( \frac{A}{B} \) valores satis cognitos involvere queat, etiamsi eos nullo adhuc modo assignare valeamus.

§ 247. Talium autem fractionum continuarum mihi quidem binæ sequentes innotuere, quarum valores commodè exhibere licet.
AD TOM. I. CAP. VIII.

\[
\begin{align*}
\frac{n+1}{3n+4} &= \frac{5n+1}{7n+1} \quad \text{et}
\frac{n+1}{9n+\text{etc.}} &= \frac{\frac{2}{e^n}}{e^n - 1},
\end{align*}
\]

Harum fractionum prior cum formulis postremi exempli collatis praebet \( m - b = n, \ 2m - b = 3n \), ideoque \( m = 2n \) et \( b = n \).

unde fit
\[
A = \frac{1}{2^n} \int \frac{\partial x}{\frac{5}{x^2} \left( 1 + x^2 \right)^{\frac{1}{2}}} e^{\frac{2\pi x}{2n}},
\]
\[
B = \frac{1}{2^n} \int \frac{\partial x}{\frac{5}{x^2} \left( 1 + x^2 \right)^{\frac{1}{2}}} e^{\frac{2\pi x}{2n}}.
\]

unde iam discimus si haec duae formae integrentur a termino \( x = 0 \) usque ad terminum \( x = \infty \), tum fore
\[
\frac{A}{B} = \frac{1 - e^n}{1 - e^n}.
\]

quannam nulla adhuc via analytica patet, hanc convenientiam demonstrandi.