(55)

ANALYSIS
FACILIS ET PLANA AD EAS SERIES MAXIME
ABSTRVSAS PERDVCENS,
QUIBVS OMNVM
AEQVATIONVM ALGEBRAICARVM
NON SOLVM RADICES IPSAE,
SED ETIAM QVAEVIS EARVM POTESTATES
EXPRIMI POSSVNT.

Auctore
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Conuent. exhib. die 15 April. 1776.

Problema.

Proposita aequatione algebraica tribus terminis constante, quam
semper hac forma representare licet: \( r = \frac{A}{x^a} + \frac{B}{x^b} \), inuenire
seriem, quae exprimat valorem ipsius \( x^n \).

Solutio.

§. 1. Haece aequatio, ponendo \( x = A^a Z \), semper ad
hanc formam simpliciorum revocari potest: \( r = \frac{A}{Z^a} + \frac{B}{A^a Z^b} \),
unde ponendo \( B = A^a C \), erit \( r = \frac{C}{Z^a} + \frac{C}{Z^b} \), quae per \( Z^a 
mul-
multiplicata praebet $Z^n = Z^{n-a} + C Z^{n-\beta}$; hinc igitur quaeo oportet valorem potestatis $Z^n$, quandoquidem hinc erit

$x^n = A^2 Z^n$. Manifestum autem est valorem ipsius $Z^n$ exprimi mi debere per seriem, in quam exponens $n$ ingreditur, quam
ergo spectare licebit tanquam functionem ipsius $n$, et quia haec
series ex infinitis terminis confabat, eam ita repraesentemus:

$$Z^n = f^0 : n + f^1 : n + f'' : n + f''' : n + \text{ etc.}$$

quae ergo forma ita debet esse comparata, vt posito $n = o$
fiat $Z^n = 1$; vnde patet statui debere $f^0 : n = 1$, reliquis ve
ro terminos factorem habere debere $n$, vt evanesceant posito
$n = o$, prodeatque $x^\infty = 1$.

§ 2. Constituta hac serie, si loco $n$ scribamus $n-a$,
habeimus:

$$Z^{n-a} = f^0 : (n-a) + f^1 : (n-a) + f'' : (n-a) + f''' : (n-a) + \text{ etc.}$$

similique modo erit

$$Z^{n-\beta} = f^0 : (n-\beta) + f^1 : (n-\beta) + f'' : (n-\beta) + \text{ etc.}$$

vbi iterum notetur esse $f^0 : (n-a) = 1$ et $f^0 : (n-\beta) = 1$.
Cum iam nostra aequatio sit $Z^n - Z^{n-a} = C Z^{n-\beta}$, scriba
mus loco potestatum ipsius $Z$ serier assumtas sequenti modo:

$$+ Z^n = f^0 : n + f' : n + f'' : n + f''' : n + \text{ etc.}$$

$$- Z^{n-a} = f^0 : (n-a) - f' : (n-a) - f'' : (n-a) - f''' : (n-a) - \text{ etc.}$$

$$\equiv C Z^{n-\beta} = C f^0 : (n-\beta) + C f' : (n-\beta) + C f'' : (n-\beta) + C f''' : (n-\beta) + \text{ etc.}$$

nunc functiones istae indefinitae ita determinatur, vt fiat

I. $f^0 : n - f^0 : (n-a) = C f^0 : (n-\beta) = C$,

II. $f' : n - f' : (n-a) = C f' : (n-\beta)$;
III. \( f^{'''}: n - f^{'''}: (n - \alpha) = C f^{'''}: (n - \beta); \)
IV. \( f^{'''}: n - f^{'''}: (n - \alpha) = C f^{'''}: (n - \beta); \)

etc.

\[ \text{(57)} \]

§ 3. Ope harum aequationum ergo primo quaerit debet natura functionis \( f': n \), vt primae aequationi satisfiat; quae inuenta innotescet function \( f': (n - \beta) \), ex eaque per secundam aequationem quaerit debet indoles functionis \( f': n \), unde innotescet function \( f^{'''}: (n - \beta) \), hincque porro simili modo ex aequatione tertia dedicatur indoles functionis \( f^{'''}: n \), et ita porro, donec lex pateat, qua singulae hae functiones viterius progressiuntur, unde paret resolutionem omnium harum aequationum revocari ad hanc quaestionem, quae proposita functione ipsius \( n \) quaeritur alia functione, veluti \( \Phi: n \), vt fiat \( \Phi: n - \Phi: (n - \alpha) = N \), quem in finem sequentia Lemmata evoluamus.

**Lemma I.**

§ 4. Si fuerit \( \Phi: n = \Delta n \), erit \( \Phi: (n - \alpha) = \Delta (n - \alpha) \);
ideoque \( \Phi: n - \Phi: (n - \alpha) = \Delta \alpha \); unde vicissime, si ponatur \( \Delta \alpha = k \), ut fieri debet \( \Phi: n - \Phi: (n - \alpha) = k \), reperietur \( \Phi: n = \frac{k n}{\alpha} \). Quare cum ex prima aequatione esse debat \( f': n - f': (n - \alpha) = C \), necesse est ut fit \( f': n = \frac{c n}{\alpha} \), unde pro secunda aequatione fiet \( f': (n - \beta) = \frac{c}{\alpha} (n - \beta) \).

**Lemma II.**

§ 5. Si fuerit \( \Phi: n = \Delta n (n + \alpha - \nu) \), erit
\( \Phi: (n - \alpha) = \Delta (n - \alpha) (n - \nu) \), unde colligitur
\( \Phi: n - \Phi: (n - \alpha) = 2 \Delta \alpha (n - \frac{1}{2} \nu) \).
Quodsi ergo prodire debet
\( \Phi: n - \Phi: (n - \alpha) = k (n - \lambda) \),
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\[ \text{H} \quad \text{ob} \]
ob \( \Delta = \frac{k}{z_a} \) et \( \nu = 2 \lambda \), erit
\[ \Phi : n = \frac{k n}{z_a} (n + \alpha - 2 \lambda). \]
Quare cum aequatio secunda iam fit
\[ f'' : n - f'' : (n - \alpha) = C f' : (n - \beta) = \frac{c c}{z_a} (n - \beta), \]
ob \( k = \frac{c c}{z_a} \) et \( \lambda = \beta \) erit
\[ f'' : n = \frac{c c}{z_a} n (n + \alpha - 2 \beta), \]
vnde pro tertia aequatione fiet
\[ f' : (n - \beta) = \frac{c c}{z_a} (n - \beta) (n + \alpha - 3 \beta). \]

**Lemma III.**

\( \S. 6. \) Si fuerit
\[ \Phi : n = \Delta n (n + \alpha - \nu) (n + 2 \alpha - \nu), \]
\[ \Phi : (n - \alpha) = \Delta (n - \alpha) (n - \nu) (n + \alpha - \nu), \]
hinc ergo fit
\[ \Phi : n - \Phi : (n - \alpha) = 3 \Delta \alpha (n + \alpha - \nu) (n - \frac{3}{2} \nu), \]
vnde vicissim, post 3 \( \Delta \alpha = k \) et \( \frac{3}{2} \nu = \lambda \), vt prodeat
\[ k (n + \alpha - 3 \lambda) (n - \lambda), \]
vnde debet
\[ \Phi : n = \frac{k n}{z_a} (n + \alpha - 3 \lambda) (n + 2 \alpha - 3 \lambda). \]

Quia nunc pro nostra aequatione tertia fieri debet
\[ f''' : n - f'' : (n - \alpha) = \frac{c \nu}{z_a} (n - \beta) (n + \alpha - 3 \beta), \]
facta aequatione fiet \( \frac{c \nu}{z_a} \) et \( \lambda = \beta \), hincque concluuntur fore
\[ f''' : n = \frac{c \nu}{z_a^3} n (n + \alpha - 3 \beta) (n + 2 \alpha - 3 \beta), \]
vnde pro aequatione sequente habebimus:
\[ f''' : (n - \beta) = \frac{c \nu}{z_a^3} (n - \beta) (n + \alpha + 4 \beta) (n + 2 \alpha - 4 \beta). \]

Lem-
Lemma IV.

§. 7. Si fuerit
\[ \Phi : n = \Delta n (n+\alpha-\nu) (n+2\alpha-\nu) (n+3\alpha-\nu), \]

erit
\[ \Phi : (n-\alpha) = \Delta (n-\alpha) (n-\nu) (n+\alpha-\nu) (n+2\alpha-\nu), \]

hincque
\[ \Phi : n - \Phi : (n-\alpha) = 4 \Delta \alpha (n+\alpha-\nu) (n+2\alpha-\nu) (n-\frac{\alpha}{2} \nu). \]
Quare si debeat esse:
\[ \Phi : n - \Phi : (n-\alpha) = k (n-\lambda) (n+\alpha-4\lambda) (n+2\alpha-4\lambda), \]

fumi debet \( \Delta = \frac{k}{\alpha} \) et \( \nu = 4 \lambda \), hincque fiet
\[ \Phi : n = \frac{k}{\alpha} (n+\alpha-4\lambda) (n+2\alpha-4\lambda) (n+3\alpha-4\lambda). \]

Quare cum aequatio nostra quarta fit
\[ f''' : n - f''' : (n-\alpha) = \frac{ct}{6 \alpha^3} (n-\beta) (n+\alpha-4\beta) (n+2\alpha-4\beta), \]

facta applicatione fiet \( k = \frac{ct}{6 \alpha^3} \) et \( \lambda = \beta \), hincque concluditur fore
\[ f''' : n = \frac{ct}{6 \alpha^3} n (n+\alpha-4\beta) (n+2\alpha-4\beta) (n+3\alpha-4\beta), \]

unde pro quinta aequatione nanciscemur:
\[ f''' : (n-\beta) = \frac{ct}{6 \alpha^3} (n-\beta) (n+\alpha-5\beta) (n+2\alpha-5\beta) (n+3\alpha-5\beta). \]

Lemma V.

§. 8. Si fuerit
\[ \Phi : n = \Delta n (n+\alpha-\nu) (n+2\alpha-\nu) (n+3\alpha-\nu) (n+4\alpha-\nu), \]

erit
\[ \Phi : (n-\alpha) = \Delta (n-\alpha) (n-\nu) (n+\alpha-\nu) (n+2\alpha-\nu) (n+3\alpha-\nu), \]

hincque
\[ \Phi : n - \Phi : (n-\alpha) = 5 \Delta \alpha (n+\alpha-\nu) (n+2\alpha-\nu) (n+3\alpha-\nu) (n-\frac{\alpha}{5} \nu). \]
Quare si debeat esse
\[ \Phi : n = 2 \Phi : n \]
\[ (60) \]

\[ \Phi : n - \Phi : (n - \alpha) = k (n - \lambda) (n + \alpha - 5 \lambda) (n + 2 \alpha - 5 \lambda) (n + 3 \alpha - 5 \lambda), \]

fumi debet \( \Delta = \frac{k}{5 \alpha} \) et \( 5 \lambda = v \), tum vero erit

\[ \Phi : n = \frac{k}{5 \alpha} n (n + \alpha - 5 \lambda) (n + 2 \alpha - 5 \lambda) (n + 3 \alpha - 5 \lambda) (n + 4 \alpha - 5 \lambda). \]

Aequatio autem quinta cum ita se habeat:

\[ \Phi'' : n - \Phi''' : (n - \alpha) = \frac{c}{2 k \alpha^2} (n - \beta) (n + \alpha - 5 \beta) (n + 2 \alpha - 5 \beta) (n + 3 \alpha - 5 \beta), \]

hic fumi debet \( k = \frac{c}{2 k \alpha^2} \) et \( \lambda = \beta \), unde concluditur

\[ \Phi''' : n = \frac{c}{180 \alpha^2} n (n + \alpha - 5 \beta) (n + 2 \alpha - 5 \beta) (n + 3 \alpha - 5 \beta) (n + 4 \alpha - 5 \beta). \]

Hinc iam fine ulteriori calculo concludere licet fore

\[ \Phi^V : n = \frac{c}{720 \alpha^2} n (n + \alpha - 6 \beta) (n + 2 \alpha - 6 \beta) (n + 3 \alpha - 6 \beta) \times \]

\[ \times (n + 4 \alpha - 6 \beta) (n + 5 \alpha - 6 \beta) \quad \text{et} \]

\[ \Phi^V : n = \frac{c}{40 \alpha^2} (n + \alpha - 7 \beta) (n + 2 \alpha - 7 \beta) (n + 3 \alpha - 7 \beta) \times \]

\[ \times (n + 4 \alpha - 7 \beta) (n + 5 \alpha - 7 \beta) (n + 6 \alpha - 7 \beta). \]

**Conclusio finalis.**

\( \S. \ 9. \) His igitur colligendis si aequatio proposta fuerit \( i = \frac{1}{Z^a} + \frac{C}{Z^b} \), tum pro potestate quacunque ipsius \( Z \) sequens resultat serie:

\[ Z^n = 1 + \frac{c}{a} n + \frac{c}{a^2} n (n + 2 \beta) + \frac{c}{a^3} n (n + \alpha - 3 \beta) (n + 2 \alpha - 3 \beta) \]

\[ + \frac{c}{a^4} n (n + \alpha - 4 \beta) (n + 2 \alpha - 4 \beta) (n + 3 \alpha - 4 \beta) \]

\[ + \frac{c^3}{180 \alpha^2} n (n + \alpha - 5 \beta) (n + 2 \alpha - 5 \beta) (n + 3 \beta - 5 \beta) \times \]

\[ \times (n + 4 \alpha - 5 \beta) + \text{etc.} \]

**Scholion.**

\( \S. \ 10. \) Haec serie, quam eruimus, eo magis est notatun digna, quod nulla alia via patet eam inuenienda. Quin etiam
iam Analytis nostra ita est comparata, vt veritas solutionis non solum ad omnes exponentes integros \( n \), sed etiam ad quosuis valores fractos, atque adeo negativos extenditur. Praeterea vero etiam ex nostra serie generali logarithmus Hyperbolicus ipsius \( Z \) exprimi potest. Cum enim semper, cæl \( n = 0 \), fit 

\[
\frac{Z^n - 1}{n} = lZ,
\]

erit nostro cæl

\[
lZ = \frac{c}{a} + \frac{c c}{a a^2} (\alpha - 2 \beta) + \frac{c^3}{a^3} (\alpha - 3 \beta) (2 \alpha - 3 \beta)
\]

\[
+ \frac{c^5}{a^5} (\alpha - 4 \beta) (3 \alpha - 4 \beta)
\]

\[
+ \frac{c^7}{a^7} (\alpha - 5 \beta) (4 \alpha - 5 \beta) + \text{etc.}
\]

\( \square \) unde si tota haec serie desingetur littera \( \Delta \), vt fit \( l'Z = \Delta \), erit \( Z = e^\Delta \), ideoque \( Z^n = e^{n \Delta} \), quae ergo quantitas aequalis erit seriei supra inuentae pro \( Z^n \). At vero ista expressio \( e^\Delta \) in seriem evoluta praebet

\[
Z^n = 1 + n \Delta + \frac{1}{2} n^2 \Delta^2 + \frac{1}{3} n^3 \Delta^3 + \frac{1}{4} n^4 \Delta^4 + \frac{1}{5} n^5 \Delta^5 + \text{etc.}
\]

quae ergo series seriei supra inuentae necessario erit aequalis, id quod etiam comprobabitur, dum saltem priores termini evoluentur. Cum enim fit

\[
\Delta = \frac{c}{a} + \frac{c c}{a a^2} (\alpha - 2 \beta) + \text{etc. erit}
\]

\[
\Delta^3 = \frac{c^3}{a^3}
\]

\[
\Delta^5 = \frac{c^5}{a^5}
\]

unde deducimus

\[
Z^n = 1 + \frac{c}{a} n + \frac{c c}{a a} (\alpha - 2 \beta) + \frac{c^3}{a^3} n (\alpha - 3 \beta) (2 \alpha - 3 \beta) + \text{etc.}
\]

\[
+ \frac{c^5}{a^5} n n (\alpha - 4 \beta) (3 \alpha - 4 \beta) + \text{etc.}
\]

\[
+ \frac{c^7}{a^7} n^3 + \text{etc.}
\]

\( \square \)
\[ Z^n = 1 + \frac{c}{a} n + \frac{c}{a^2} n (n + \alpha - 2 \beta) + \frac{c^3}{a^3} (n + \alpha - 3 \beta) (n + 2 \alpha - 3 \beta) + \ldots \]

quod cum serie pro \( Z^n \) supra inuenta perfecte congruit.

**Theorema generale.**

\[ \text{§. xi. Quodsi ergo proposita fuerit aequatio initio commemorata: } 1 = \frac{A}{x^a} + \frac{B}{x^b}, \text{ quoniam posuimus } Z = \frac{x}{A^2} \text{ et} \]

\[ C = \frac{B}{A^2}, \text{ erit} \]

\[ \frac{x^n}{A^a} = 1 + \frac{B}{A^a} n + \frac{B^2}{A^a} \frac{n(n + \alpha - 2 \beta)}{2 \alpha} \]

\[ + \frac{B^3}{A^a} \frac{n(n + \alpha - 3 \beta)(n + 2 \alpha - 3 \beta)}{3 \alpha} \]

\[ + \frac{B^4}{A^a} \frac{n(n + \alpha - 4 \beta)(n + 2 \alpha - 4 \beta)(n + 3 \alpha - 4 \beta)}{4 \alpha} \text{ etc.} \]

\[ \text{flue:} \]

\[ X^n = A^a + A^a \frac{B}{\alpha} n + A^a \frac{B^2}{\alpha} \frac{n(n + \alpha - 2 \beta)}{2 \alpha} \]

\[ + A^a \frac{B^3}{\alpha} \frac{n(n + \alpha - 3 \beta)(n + 2 \alpha - 3 \beta)}{3 \alpha} \]

\[ + A^a \frac{B^4}{\alpha} \frac{n(n + \alpha - 4 \beta)(n + 2 \alpha - 4 \beta)(n + 3 \alpha - 4 \beta)}{4 \alpha} \]

\[ + A^a \frac{B^5}{\alpha} \frac{n(n + \alpha - 5 \beta)(n + 2 \alpha - 5 \beta)(n + 3 \alpha - 5 \beta)}{5 \alpha} \times \]

\[ + \text{ etc.} \]
Vicissim igitur proposita hac serie, eius summa erit \( x^n \), existente \( x \) radice huius aequationis: \( x = \frac{A}{x^a} + \frac{B}{x^b} \), id quod aliquot exemplis illustrare liceat.

**Exemplum 1.**

§. 12. Statuamus \( \alpha = x \) et \( \beta = x \), vt proposita sit ista aequatio: \( x = \frac{A}{x^a} + \frac{B}{x^b} \), unde sit \( x = A + B \), consequenter \( x^n = (A + B)^n \), series autem invenita hoc casu nobis dat

\[
(A + B)^n = A^n + \frac{n}{1} A^{n-1} B + \frac{n(n-1)}{1 \cdot 2} A^{n-2} B^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} A^{n-3} B^3 + \text{etc.}
\]

quae est ipsa evolutio Binomii Newtoniana, quam nunc patet veram esse, quicumque numerus pro exponente \( n \) accipiatur, sine integer, sine fractus, sine positius, sine negativus, sine etiam furtus; cum in Algebra communi, vbi haec evolutio est tractata, exponens \( n \) necessario sit integer positius.

**Exemplum 2.**

§. 13. Ponamus, vt ante, \( \alpha = x \), at sumatur \( \beta = 0 \), ita vt sit \( x = \frac{A}{x^a} + B \), unde sit \( x = \frac{A}{x - B} \), consequenter

\[
x^n = \frac{A^n}{(x - B)^n} = A^n (x - B)^{-n},
\]

series autem, ad quem sumus perducit, hoc casu erit

\[
A^n (x - B)^n = A^n + \frac{n}{1} A^{n-1} B + \frac{n(n+1)}{1 \cdot 2} A^{n-2} B^2 + \text{etc.}
\]

sine

\[
(x - B)^n = x + \frac{n}{1} B + \frac{n(n+1)}{1 \cdot 2} B^2 + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} B^3 + \text{etc.}
\]

quae est demonstratio ciusdem theoromatis Newtoniani pro exponentibus negativis.
Exemplum 3.

§. 14. Sumamus \( A = 2a \) et \( B = b \), itaque porro \( a = x \) et \( \beta = 2 \), vt nostra aequatio fiat \( x = \frac{3a}{x^2} + \frac{b}{x^2} \), fiue \( xx = 2ax + b \), vnde fit \( x = a + \sqrt{a^2 + b} \), quo valore substituto series ante inuenta praebebit

\[
(a + \sqrt{a^2 + b})^n = a^n + \frac{n}{2} a^{n-2} b + \frac{n(n-2)}{3} a^{n-4} b^2 + \frac{n(n-2)(n-4)}{4} a^{n-6} b^3 + \frac{n(n-2)(n-4)(n-6)}{5} a^{n-8} b^4 + \text{etc.}
\]

cuius veritas pro casso, quo \( n = x \), ex evoluzione vulgari confirmari potest. Sumto enim \( n = x \) erit

\[
a + \sqrt{a^2 + b} = 2a + \frac{b}{a} - \frac{b}{2a} + \frac{b}{3a^2} - \frac{b}{4a^3} + \text{etc.}
\]

nouimus autem ex resolutione vulgari esse

\[
\sqrt{(a^2 + b)} = a + \frac{b}{2a} - \frac{b}{4a^2} + \frac{b}{6a^3} - \frac{b}{8a^4} + \text{etc.}
\]

cui si addatur \( a \), ipsa illa series prodit.

Exemplum 4.

§. 15. Sumamus \( a = 2 \) et \( \beta = x \), vt fit \( x = \frac{A}{x} + \frac{B}{x} \), fiue \( xx = A + B \), ideoque \( x = \frac{b + \sqrt{(b^2 + 4a)}}{a} \), loco \( A \) autem scribamus \( a^2 \) et \( 2b \) loco \( B \), vt fit \( x = b + \sqrt{(b^2 + 4a)} \), quocirca series inuenta nobis dabat

\[
(b + \sqrt{b^2 + 4a})^n = a^n + \frac{n}{2} a^{n-2} b + \frac{n(n-2)}{3} a^{n-4} b^2 + \frac{n(n-2)(n-4)}{4} a^{n-6} b^3 + \frac{n(n-2)(n-4)(n-6)}{5} a^{n-8} b^4 + \text{etc.}
\]

quae reductur ad hanc formam:

\[
(b + \sqrt{b^2 + 4a})^n = a^n + \frac{n}{2} a^{n-2} b + \frac{n(n-2)}{3} a^{n-4} b^2 + \frac{n(n-2)(n-4)}{4} a^{n-6} b^3 + \text{etc.}
\]

haec autem forma viterius reductur ad hanc:

\[
(b + \sqrt{b^2 + 4a})^n = a^n + \frac{n}{2} a^{n-2} b + \frac{n(n-2)}{3} a^{n-4} b^2 + \frac{n(n-2)(n-4)}{4} a^{n-6} b^3 + \text{etc.}
\]
$$\begin{align*}
\left(b + \sqrt{b^2 + a^2}\right)^n &= a^n + \frac{n}{2} a^{n-1} b + \frac{\binom{n}{3}}{6} a^{n-2} b^2 \\
&+ \frac{\binom{n}{4}}{24} a^{n-3} b^3 + \frac{\binom{n}{5}}{120} a^{n-4} b^4 + \frac{\binom{n}{6}}{720} a^{n-5} b^5 \\
&+ \frac{\binom{n}{7}}{5040} a^{n-6} b^6 + \frac{\binom{n}{8}}{40320} a^{n-7} b^7 \\
&+ \text{ etc.}
\end{align*}$$

Ita si sumamus \( n = 1 \), habebimus
$$b + \sqrt{b^2 + a^2} = a + b + \frac{1}{2} \frac{b}{a} - \frac{3}{4} \frac{b^2}{a^2} + \frac{5}{6} \frac{b^3}{a^3} - \text{ etc.}$$

nouimus autem esse
$$\sqrt{b^2 + a^2} = a + \frac{1}{2} \frac{b}{a} - \frac{3}{4} \frac{b^2}{a^2} + \frac{5}{6} \frac{b^3}{a^3} - \text{ etc.}$$
cui si addatur \( b \), ipsa illa serie prodit.

**Exemplum 5.**

§. 16. Sumamus \( \alpha = -1 \) et \( \beta = x_1 \), vt nostra accuratio sit \( x = \frac{x_1 - 4 \alpha \beta}{a \beta} \), hinc ergo prodit
$$\left(1 + \sqrt{1 - 4 \alpha \beta}\right)^n = A^n + \frac{n}{2} A^{n-1} B + \frac{\binom{n}{3}}{6} A^{n-2} B^2$$
$$+ \frac{\binom{n}{4}}{24} A^{n-3} B^3 + \frac{\binom{n}{5}}{120} A^{n-4} B^4$$
$$+ \frac{\binom{n}{6}}{720} A^{n-5} B^5 + \text{ etc.}$$

Hinc ergo si sumamus \( n = 1 \), erit
$$\frac{x_1 + \sqrt{1 - 4 \alpha \beta}}{a \beta} = A + A^2 B + \frac{A^3}{3} B^2 + \frac{A^4}{6} B^3$$
$$+ \frac{7}{2} \frac{A^5}{3} B^4 + \frac{19}{2} \frac{A^6}{4} B^5 + \text{ etc.}$$

Eft vero
$$\sqrt{(1 - 4 \alpha B)} = 1 - A \beta - 2 A^2 \beta - 4 A^3 \beta$$
$$- 2 A^4 \beta - \text{ etc.}$$

quae serie ab unitate subtrahit et per \( 2B \) dividitur praebet serie modo inuentam.

*Noua Acta Acad. Imp. Sc. T. IV.*

**I**

Scho-
§. 17. Series autem generalis, quam supra eliciimus, primum ab acutissimo Lamberto ex principiis maxime dieresis est inuenta, quam idcirco Lamberinam appellare liceat, prop- terea quod inter egregia huius viri inuenta merito est referen- da. Methodus autem, qua hic vi fumus, ad aequationes mul- to generaliores extendi potest, quando feliciter aequatio pro- posita quattuor pluresque terminos continet; id quod pro cauf quattuor terminorum offendi esse operae erit pretium.

Problema generalius.

Si proposta fuerit aequatio algebraica huius formae:

\[ \frac{n}{Z^n} = \frac{B}{Z^\beta} + \frac{C}{Z^\gamma}, \]

inuenire seriem, quae valorem potestatis cuiuscunque ipsius Z, puta Z^n exprimat.

Solutio.

§. 18. Multiplicetur aequatio proposta per Z^n, ut habeatur Z^n = Z^n-\alpha = B Z^n-\beta + C Z^n-\gamma; et posteaatem quae- fitam Z^n ut ante tanquam functionem ipsius n spectare licebit, quae per partes continuo procedentes ita repraesentetur, ut fit

\[ Z^n = f^0 : n + f' : n + f'' : n + f''' : n + \text{ etc.} \]

ubi cum sumto n = o fieri debet Z^n = 1, fit perpeta f^0 : n = 1, tum vero ut reliqua partes casu n = 0 evanescant, finguas factorem n habere necesse est. Hinc ergo erit

\[ Z^n-\alpha = f^0 : (n - \alpha) + f' : (n - \alpha) + f'' : (n - \alpha) + \text{ etc.} \]
\[ Z^n-\beta = f^0 : (n - \beta) + f' : (n - \beta) + f'' : (n - \beta) + \text{ etc.} \]
\[ Z^n-\gamma = f^0 : (n - \gamma) + f' : (n - \gamma) + f'' : (n - \gamma) + \text{ etc.} \]

Iam
Iam istae series loco harum potestatum in nostra aequatione substituuntur, et cum partes in membro sinistro sponte se tollant, reliquae partes sinistrae partibus antecedentibus in dextro membro aequari debentur, unde sequentes aequationes refutabunt.

I. \( f' : n - f' : (n - \alpha) = B f' : (n - \beta) + C f' : (n - \gamma) = B + C \)

II. \( f'' : n - f'' : (n - \alpha) = B f'' : (n - \beta) + C f' : (n - \gamma) \)

III. \( f''' : n - f''' : (n - \alpha) = B f''' : (n - \beta) + C f'' : (n - \gamma) \)

IV. \( f'''' : n - f'''' : (n - \alpha) = B f'''' : (n - \beta) + C f''' : (n - \gamma) \)

etc.

§. 19. In subsidium iam vocemus lemmata supra allata, ex quibus conflat fore vt sequitur:

I. Si fuerit \( \Phi : n - \Phi : (n - \alpha) = k \), erit \( \Phi : n = \frac{k n}{\alpha} \),

II. Si fuerit \( \Phi : n - \Phi : (n - \alpha) = k (n - \lambda) \), erit

\( \Phi : n = \frac{k n}{\alpha} (n + \alpha - 2 \lambda) \).

III. Si fuerit \( \Phi : n - \Phi : (n - \alpha) = k (n - \lambda) (n + \alpha - 3 \lambda) \), erit

\( \Phi : n = \frac{k n}{\alpha} (n + \alpha - 3 \lambda) (n + 2 \alpha - 3 \lambda) \).

IV. Si fuerit \( \Phi : n - \Phi : (n - \alpha) = k (n - \lambda) (n + \alpha - 4 \lambda) (n + 2 \alpha - 4 \lambda) \),

erit

\( \Phi : n = \frac{k n}{\alpha} (n + \alpha - 4 \lambda) (n + 2 \alpha - 4 \lambda) (n + 3 \alpha - 4 \lambda) \).

V. Si fuerit \( \Phi : n - \Phi : (n - \alpha) = k (n - \lambda) (n + \alpha - 5 \lambda) (n + 2 \alpha - 5 \lambda) (n + 3 \alpha - 5 \lambda) \),

erit

\( \Phi : n = \frac{k n}{\alpha} (n + \alpha - 5 \lambda) (n + 2 \alpha - 5 \lambda) (n + 3 \alpha - 5 \lambda) (n + 4 \alpha - 5 \lambda) \),

et ita porro.

I 2

§. 20.
§. 20. Harum lemmatum őpe ex prima aequatione, ubi pro lemmate primo est \( k = B + C \), elicitus \( f': n = \frac{B n}{a} + \frac{C n}{a} \) hinc igitur pro secunda aequatione erit

\[
B f' : (n - \beta) = B^a \left( \frac{n - \beta}{a} \right) + B C \left( \frac{n - \beta}{a} \right) \quad \text{et}
\]

\[
C f' : (n - \gamma) = C^a \left( \frac{n - \gamma}{a} \right) + B C \left( \frac{n - \gamma}{a} \right),
\]

Summa

\[
= \frac{B^a (n - \beta) + \frac{B C}{a} (n - \beta - \gamma)}{a} + \frac{C^a (n - \gamma)}{a} + \frac{C n}{a} (n - \gamma),
\]

quae formula quia ex tribus confat partibus, singulas cum lemmate secundo conferri oportet, ac pro prima parte erit \( k = \frac{B n}{a} \) et \( \lambda = \beta \), pro secunda parte erit \( k = \frac{B C}{a} \) et \( \lambda = \frac{\beta - \gamma}{a} \), pro tertia parte erit \( k = \frac{C n}{a} \) et \( \lambda = \gamma \), unde ex omnibus fitum sumitur:

\[
f'' : n = \frac{B n}{a} (n + \alpha - 2 \beta) + \frac{B C}{a} (n (n + \alpha - \beta - \gamma))
\]

\[
\quad + \frac{C n}{a} (n + \alpha - 2 \gamma).
\]

§. 21. Progrediamur iam ad aequationem tertiam, ac pro eius membro dextro habebimus:

\[
B f'' : (n - \beta) = \frac{B n}{a} (n - \beta) (n + \alpha - 3 \beta)
\]

\[
+ \frac{B B C}{a^2} (n - \beta) (n + \alpha - 2 \beta - \gamma) + \frac{B C^2}{a^2} (n - \beta) (n + \alpha - \beta - 2 \gamma),
\]

\[
C f'' : (n - \gamma) = \frac{C n}{a} (n - \gamma) (n + \alpha - 3 \gamma)
\]

\[
+ \frac{B B C}{a^2} (n - \gamma) (n + \alpha - 3 \beta - \gamma) + \frac{C^2}{a^2} (n - \gamma) (n + \alpha - 2 \beta - 2 \gamma)
\]

Summa

\[
= \frac{B n}{a} (n + \alpha - 3 \beta)
\]

\[
+ \frac{3 B n C}{a^2} (n + \alpha - 2 \beta - \gamma) (n - \frac{\beta - \gamma}{3})
\]

\[
+ \frac{3 B C}{a^2} (n + \alpha - \beta - 2 \gamma) (n - \frac{\beta - 2 \gamma}{3})
\]

\[
+ \frac{C n}{a} (n - \gamma) (n + \alpha - 3 \gamma),
\]

quae
quae quia quatuor confat partibus, cum lemmate III. confe-
rendis, pro prima parte erit \( k = \frac{\beta^2}{6a^3} \) et \( \lambda = \beta \), pro secunda
parte erit \( k = \frac{3\beta c}{6a^3} \) et \( \lambda = \frac{\beta + \gamma}{3} \); pro tertia vero parte est
\( k = \frac{3b c}{6a^3} \) et \( \lambda = \frac{\beta + \gamma}{3} \), denique pro quarta parte est \( k = \frac{\beta}{8a^3} \)
et \( \lambda = \gamma \), quibus obseruatis functio quaestita \( f'''' \) itidem ex
quatuor partibus confabat, quae sunt:

\[
f'''' : n = \left\{ \begin{array}{l}
 + \frac{2b^2}{6a^3} n (n + \alpha - 3 \beta) (n + 2 \alpha - 3 \beta) \\
 + \frac{3b c}{6a^3} n (n + \alpha - 2 \beta - \gamma) (n + 2 \alpha - 2 \beta - \gamma) \\
 + \frac{3n c}{6a^3} n (n + \alpha - \beta - 2 \gamma) (n + 2 \alpha - \beta - 2 \gamma) \\
 - \frac{c^2}{6a^3} n (n + \alpha - 3 \gamma) (n + 2 \alpha - 3 \gamma).
\end{array} \right.
\]

\[\text{§ 22. Tractemus similis modo aequationem quartam,}
\]
aque ex valore \( f'''' : n \) inuenito habebimus:

\[
B f'''' : (n - \beta) = \left\{ \begin{array}{l}
 + \frac{b^2}{6a^3} (n - \beta) (n + \alpha - 4 \beta) (n + 2 \alpha - 4 \beta) \\
 + \frac{b^2 c}{6a^3} (n - \beta) (n + \alpha - 3 \beta - \gamma) (n + 2 \alpha - 3 \beta - \gamma) \\
 + \frac{3b c}{6a^3} (n - \beta) (n + \alpha - 2 \beta - 2 \gamma) (n + 2 \alpha - 2 \beta - 2 \gamma) \\
 + \frac{c^2}{6a^3} (n - \beta) (n + \alpha - \beta - 3 \gamma) (n + 2 \alpha - \beta - 3 \gamma)
\end{array} \right.
\]

\[
C f'''' : (n - \gamma) = \left\{ \begin{array}{l}
 + \frac{b^2 c}{6a^3} (n - \gamma) (n + \alpha - 3 \beta - \gamma) (n + 2 \alpha - 3 \beta - \gamma) \\
 + \frac{3b c}{6a^3} (n - \gamma) (n + \alpha - 2 \beta - 2 \gamma) (n + 2 \alpha - 2 \beta - 2 \gamma) \\
 + \frac{c^2}{6a^3} (n - \gamma) (n + \alpha - 4 \gamma) (n + 2 \alpha - 4 \gamma)
\end{array} \right.
\]

His iam terminis collectis valor formulæ

\[B f'''' : (n - \beta) + C f'''' : (n - \gamma),\]

constat in sequentibus quinque partibus:

\[13\]

\[B^*\]
\[
\frac{\mathfrak{b}_4}{\mathfrak{a}_3} \left( n - \beta \right) \left( n + a - 4 \beta \right)
\]
\[
+ \frac{\mathfrak{b}_4 \mathfrak{c}_2}{\mathfrak{a}_3} \left( n - \frac{3 \beta - \gamma}{4} \right) \left( n + a - 3 \beta - \gamma \right) \left( n + 2 a - 3 \beta - \gamma \right)
\]
\[
+ \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \left( n - \beta - \gamma \right) \left( n + a - 2 \beta - 2 \gamma \right) \left( n + 2 a - 2 \beta - 2 \gamma \right)
\]
\[
+ \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \left( n - \frac{\beta - 3 \gamma}{4} \right) \left( n + a - \beta - 3 \gamma \right) \left( n + 2 a - \beta - 3 \gamma \right)
\]
\[
+ \frac{\mathfrak{c}_4}{\mathfrak{a}_3} \left( n - \gamma \right) \left( n + a - 4 \gamma \right) \left( n + 2 a - 4 \gamma \right)
\]

§. 23. Quoniam igitur functio quae sita \( f^{IV} : n \) ex quinque partibus componitur, singulas cum lemmate quarto comparari oportebit, ac pro parte prima erit \( k = \frac{\mathfrak{b}_4}{\mathfrak{a}_3} \) et \( \lambda = \beta \); pro parte secunda est \( k = \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \) et \( \lambda = \frac{3 \beta + \gamma}{4} \); pro parte tertia \( k = \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \) et \( \lambda = \frac{\beta + 3 \gamma}{4} \); pro parte quarta est \( k = \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \) et \( \lambda = \gamma \); denique pro parte quinta erit \( k = \frac{\mathfrak{c}_4}{\mathfrak{a}_3} \) et \( \lambda = \gamma \); unde collectis omnibus terminis repertiur

\[
\begin{align*}
&\left( \frac{\mathfrak{b}_4}{\mathfrak{a}_3} \left( n + a - 4 \beta \right) \left( n + 2 a - 4 \beta \right) \left( n + 3 a - 4 \beta \right) \right) \\
+ &\left( \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \left( n + a - 3 \beta - \gamma \right) \left( n + 2 a - 3 \beta - \gamma \right) \left( n + 3 a - 3 \beta - \gamma \right) \right)
\end{align*}
\]

\( f^{IV} : n = \)

\[
\begin{align*}
+ &\left( \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \left( n + a - 2 \beta - 2 \gamma \right) \left( n + 2 a - 2 \beta - 2 \gamma \right) \left( n + 3 a - 2 \beta - 2 \gamma \right) \right) \\
+ &\left( \frac{\mathfrak{b}_2 \mathfrak{c}_2}{\mathfrak{a}_3} \left( n + a - \beta - 3 \gamma \right) \left( n + 2 a - \beta - 3 \gamma \right) \left( n + 3 a - \beta - 3 \gamma \right) \right) \\
+ &\left( \frac{\mathfrak{c}_4}{\mathfrak{a}_3} \left( n + a - 4 \gamma \right) \left( n + 2 a - 4 \gamma \right) \left( n + 3 a - 4 \gamma \right) \right)
\end{align*}
\]

§. 24. Superfluum foret hos calculos viliterius proficui; quandoquidem ex allatis iam tuto concludere licet, sequentem functionem \( f^{V} : n \) hunc habituram esse valorem:
\[
F : n = \begin{cases}
+ \frac{B^3}{120 \pi^3} n(n+\alpha-5\beta)(n+2\alpha-5\beta)(n+3\alpha-5\beta)(n+4\alpha-5\beta) \\
+ \frac{B^3 C^5}{120 \pi^3} n(n+\alpha-4\beta-\gamma)(n+2\alpha-4\beta-\gamma)(n+3\alpha-4\beta-\gamma) \times (n+4\alpha-4\beta-\gamma) \\
+ \frac{10 B^3 C^5}{120 \pi^3} n(n+\alpha-3\beta-2\gamma)(n+2\alpha-3\beta-2\gamma)(n+3\alpha-3\beta-2\gamma) \times (n+4\alpha-3\beta-2\gamma) \\
+ \frac{10 B^3 C^5}{120 \pi^3} n(n+\alpha-2\beta-3\gamma)(n+2\alpha-2\beta-3\gamma)(n+3\alpha-2\beta-3\gamma) \times (n+4\alpha-2\beta-3\gamma) \\
+ \frac{5 B C^4}{120 \pi^3} n(n+\alpha-\beta-4\gamma)(n+2\alpha-\beta-4\gamma)(n+3\alpha-\beta-4\gamma) \times (n+4\alpha-\beta-4\gamma) \\
+ \frac{C^5}{120 \pi^3} n(n+\alpha-5\beta)(n+2\alpha-4\beta)(n+3\alpha-5\beta)(n+4\alpha-5\beta)
\end{cases}
\]

unde formatio omnium sequentium functionum fatis dilucide perspicitur.

§. 25. Quodsi iam omnes iti valores, quos pro functionibus \( f' : n; f'' : n; f''' : n; \) etc. elicuimus, in vnam sum-mam colligantur, et ob \( f^o : n = 1 \) vnitas praesigatur, obtinebitur serie desiderata, quae scilicet valorem possestat \( Z \) ex-primit, neque ergo opus est omnes istas functiones hic denuo collectas referre.

Corollarium.

Hae ergo series infinitis constant terminis, in quibus omnes posibiles binarum litterarum \( B \) et \( C \) combinationes occurrunt. Quin etiam pro combinatione quacunquae, quae sit \( B^3 C^5 \), in genere terminus eam inuolvens assignari poterit. Primom enim ista forma multiplicetur per numerum omnium combinationum, qui posito breuietratis gratia \( b + c = i \), si indicetur littera \( N \), erit vti constat \( N = \frac{1 \times 3 \times 3 \times \ldots \times 1}{1 \times 2 \times 3 \times \ldots \times b \times 2 \times \ldots \times c} \), dein de si ponamus \( b \beta + c \gamma = 0 \), erit terminus huic formae respondens:

\[N B^3\]
\[ (\gamma^2) \]

\[
\frac{NB^bC^e}{i \cdot 2 \cdot 3 \ldots \cdot i \alpha^2} \cdot n(n + \alpha - \theta)(n + 2\alpha - \theta) \times \\
\times (n + 3\alpha - \theta) \ldots \ldots \left[n + (i - 1)\alpha - \theta\right].
\]

Veluti sit forma proposta fuerit \( B^B \ C^C \), erit \( i = 5 \), hincque \( N = \frac{i \cdot 2 \cdot 3 \cdot 4 \cdot 5}{i \cdot 2 \cdot 3 \cdot 4} = 10 \), deinde vero erit \( \theta = 3\beta + 2\gamma \), sicque

ipse terminus huius formae erit

\[
\frac{10 \times 5 \times C^e}{125 \times \alpha^2} \cdot n(n + \alpha - 3\beta - 2\gamma)(n + 2\alpha - 3\beta - 2\gamma) \times \\
\times (n + 3\alpha - 3\beta - 2\gamma)(n + 4\alpha - 3\beta - 2\gamma),
\]

profusus ut supra est exhibitus.

Scholion.

§ 26. Hinc iam abunde patet, si aequatio proposta pluribus adhuc constet terminis, habeatque hanc formam:

\[
i = \frac{1}{Z^a} = \frac{B}{Z^B} + \frac{C}{Z^C} + \frac{D}{Z^D} + \frac{E}{Z^E} + \text{etc.}
\]

Tum ope eiusdem methodi seriem infinitam inuestigari possit, quae valorem potestatis \( Z^n \) exprimat; ista enim series incipiens ab unitate infinitis inuoluet terminos ex omnibus plane combinationibus litterarum \( B, C, D, E \), etc. formatos. Si enim in genere proponatur haec combinatio: \( B^B \ \ C^C \ \ D^D \ \ E^E \), etc. facturatur primo \( b + c + d + e = i \), et quaeatur numerus \( N \), ut sit

\[
N = \frac{i \cdot 2 \cdot 3 \cdot 4 \ldots \cdot i}{i \cdot 2 \cdot 3 \ldots \cdot i},
\]

sicque breuitatis gratia \( i \cdot 2 \cdot 3 \cdot 4 \ldots \cdot i = 1 \), factor prior huius termini erit \( \frac{NB^b \ C^C \ D^D \ E^E}{1 \ \alpha^2} \), praelucra vero adiungendi sunt factores exponentem \( n \) inuoluentes, pro quibus inueniendis facturatur:

\[
b\beta + c\gamma + d\delta + e\epsilon = \theta,
\]

\text{crunt-}
eruntque iti factores numero i iti:

\[ n \cdot (n + a - \theta) \cdot (n + 2a - \theta) \cdot \ldots \cdot [n + (i - 1) \cdot a - \theta], \]

ita vt totus terminus fit

\[
\frac{N.B^b \cdot C^c \cdot D^d \cdot E^e \cdot 1 \cdot a^i}{n \cdot (n + a - \theta) \cdot (n + 2a - \theta) \cdot \ldots \cdot [n + (i - 1) \cdot a - \theta].}
\]

Quamobrem super indole omnium harum serierum maxime memorabilium, quas olim in Tomo XV. Notorum Commentariusius fuerit, sed sufficiente demonstratione descripsi, nihil amplius desiderari posse videtur.