DE MENSURA ANGULORUM SOLIDORUM.

Auctore

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§ 1.

Quemadmodum anguli plani mensurantur per arcus circulares eos subtendentes, si scilicet vertex anguli in centro circuli collocaetur: ita naturae rei consentanei videntur, angulos solidos per portiones superficiei sphaericae metiri, quae eos quasi subtendant, si vertex anguli in centro sphaerae collocaetur. Ita si angulus solidus ex tribus angulis, qui sint \( a, b, c \), fuerit formatus, et circa verticem sphaerae describatur, cuius radius vnitate exprimatur, mensura huius anguli solidi rite statuetur areae trianguli sphaericī aequalis, cuius latera sint illis angulis \( a, b, c \) aequalia; quandoquidem haec latera sunt mensurae ilorum angulorum planorum. Eodem modo si angulus solidus ex quattuor vel pluribus angulis planis fuerit formatus, cuius mensura erit areae quadrilateri sphaericī, vel polygoni plurium laterum, cuius scilicet singula latera aequantur angulis planis, quibus angulus solidus componitur. Hac igitur ratione dimensio angulorum solidorum reducitur ad inexactitionem arcae trianguli sphaericī, vel polygoni plurium late-
laterum, cuivs latera fuerint data. Cum igitur area cuivsque trianguli sphærici facilius ex eis angulis cognoscatur, quemadmodum iam dudum al acutissimo Geometra Alberto Girardo est demonstratum, hanc ipsam demonstrationem, quoniam non minus fatis nota videtur, hic apponam.

Lemma.

§ 2. Area portionis sphæricæ, inter duos meridianos, angulo a inuicem inclinatos, contenta, se habet ad superficiem totius sphærae, ut angulus a ad 360°. Sint A C B et A D B duo semicirculi maximi in superficie sphærica, se mutuo in polis opposti A et B foceantès, et inuicem inclinati angulo C A D vel C B D = α, et eundem eff, aréam huius sectoris sphærici A C B D A toties contineri in superficie sphærae tota, quoties angulus a continetur in 360° gradibus.

§ 3. Quod si ergo radius sphærae ponatur = r, quia superficies totius sphærae est = 4 π r², denota aeté π peripheriam circuli, cuius diameter = r, erit area nöhri sectoris sphærici = 4 π r² sin² α; si quidem angulus a in gradibus exprimatur, at si α detur in partibus radii, qui semper vnitate exprimatur, ob 360° = 2 π erit area sectoris sphærici = 2 α r²; vnde si radius sphærae pariter vnitate aequalis statuatur, ista area erit = 2 a. Hodie igitur modo area illius sectoris per simplicem angulum representa potest, dum tota superficies est = 4 π.
Theorema

Alberti Girardi.

§. 4. Area trianguli sphærici semper aequalis est angulo, quo summa omnium trium angulorum trianguli sphærici exedit duos angulos rectos.

Demonstratio.

Sit ABC triangulum sphæricum propositum, cujus area quaeritur, einsque anguli denotentur litteris α, β, γ. Iam primo latera AB et AC in superficie sphærica producantur, donec fibi mutuo iterum occurrant in polo a, ipsi angulo A opposito, et quia hi arcus A B a et A C a tanquam duo meridiani spectari possunt, a se innice angulus a distant, erit area itius sectoris A C A B = 2 α. Deinde eodem modo bina latera BA et BC continuentur vsque in b, quod punctum iudem erit polus, ipsi B oppositus; huiusque sectoris B A B C area erit = 2 β. Denique producantur etiam latera CA et CB vsque in polem ipsi C oppositum in c, eritque itius sectoris C B C A area = 2 γ. Hinc igitur si area trianguli A B C quaestis vocetur = S, innotescat areae sequentium triangulorum:

I. a B C = 2 α - S
II. b A C = 2 β - S
III. c A B = 2 γ - S.

§. 4. Quia nunc, puncta a, b, c in superficie sphæricæ punctis A, B et C e diametro sunt opposita, inter se etiam easdem tenebunt distantis, etiam si in figura longe aliter videatur. Hinc dicitur arcus a b, b c, c a, erit

^Meta Acad. Imp. Sc. Tom. II. P. II. E ab =
$ab = AB$, $ac = AC$ et $bc = BC$; unde et huius trianguli $abc$, in regione sphæae posteriori fit, area quoque erit $S$; ita vt iam tota superficies sphææ continet $x^o$. triangula $ABC = S$ et $abc = S$; $2^o$. triangula $ABC = 2\alpha - S$, $bAC = 2\beta - S$ et $cAB = 2\gamma - S$.

Praeterea vero figura continet triangula $abc$, $acb$ et $bca$, quorum posteriorum areae ex superioribus innomentum; namque pro triangulo $abc$ primo est latus $ab = AB$, latus $ac = AC$ et $bc = BC$; unde manifesto hoc triangulum $abC = ABC = 2\gamma - S$. Eodem modo intelli- tur fore triangulum $acB = ACB = 2\beta - S$; ac denique $bca = BCA = 2\alpha - S$.

§ 5. Quare cum tota sphææ superficies hinc disseccta sit in octo triangula, quorum singulorum areas hic exhibuimus, earum summa aequalis esse debet toti superficie sphææ $= 4\pi$; ex qua aequalitate area quae- sita $S$ definiri poterit. Singula igitur haec triangula cum suis areis conspectui exponamus:

<table>
<thead>
<tr>
<th>I. $ABC = S$</th>
<th>III. $aBC = 2\alpha - S$</th>
<th>VI. $abc = 2\alpha - S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II. $abc = S$</td>
<td>IV. $bAC = 2\beta - S$</td>
<td>VII. $Bac = 2\beta - S$</td>
</tr>
<tr>
<td>V. $cAB = 2\gamma - S$</td>
<td>VIII. $Cab = 2\gamma - S$</td>
<td></td>
</tr>
</tbody>
</table>

Summa $= 2S + 2(\alpha + \beta + \gamma) - 3S + 2(\alpha + \beta + \gamma) - 3S$

unde omnium octo triangulorum summa colligitur $= 4(\alpha + \beta + \gamma) - 4S$, quae ergo aequalis esse debet $4\pi$, unde per quattuor dividendo ortur $\alpha + \beta + \gamma - S = \pi$; ideoque $S = \alpha + \beta + \gamma - \pi$, ubi $\alpha + \beta + \gamma$ est summa omnium angulorum trianguli propofiri, et $\pi$ est mensura duorum rectorum, sicue $180^o$, fique area trianguli sphærici.
ci propositi reperitur, si a summa omnium angulorum \( \alpha + \beta + \gamma \) duo recti seu \( x80^\circ \) subtrahantur, prorsus ut Theorema declarat.

§. 6. Totum ergo negotium pro mensura angulorum solidorum huc reducitur: vt ex datis ternis lateribus trianguli sphærici eius area definatur; quamobrem sequens Problema resoluendum fuscipiamus.

**Problema generale.**

*Datis in triangulo sphærico ternis lateribus \( AB = c \), Tab. II. AC = b et BC = a, investigare aream huius trianguli sphærici.*

**Solutio.**

§. 7. Denotent literæ \( A, B, C \) angulos huius trianguli, ponaturque eiusmod area quam quaerimus \( = S \), ac modo vidimus fore \( S = A + B + C - x80^\circ \). Hinc ergo crit fin. \( S = -\text{fin.} (A + B + C) \) et \( \text{cof.} S = -\text{cof.} (A + B + C) \), hincque tang. \( S = +\text{tang.} (A + B + C) \); sicque tantum opus est, vt loco angulorum \( A, B, C \) latera \( a, b, c \) in calculum introducantur. At vero per præcepta trigonometriae sphæricæ anguli ex datis lateribus ita definuntur, vt sit:

\[
\text{cof.} A = \frac{\text{cof.} a - \text{cof.} b - \text{cof.} c}{\text{fin.} a \text{fin.} b \text{fin.} c}; \quad \text{cof.} B = \frac{\text{cof.} b - \text{cof.} a - \text{cof.} c}{\text{fin.} a \text{fin.} b \text{fin.} c}; \\
\text{cof.} C = \frac{\text{cof.} c - \text{cof.} a - \text{cof.} b}{\text{fin.} a \text{fin.} b \text{fin.} c};
\]

vnde porro deducuntur sinus corundem angulorum

\[
\text{fin.} A = \sqrt{(1 - \text{cof.} a^2 - \text{cof.} b^2 - \text{cof.} c^2 + 2 \text{cof.} a \text{cof.} b \text{cof.} c)}; \\
\text{fin.} B = \sqrt{(1 - \text{cof.} a^2 - \text{cof.} b^2 - \text{cof.} c^2 + 2 \text{cof.} a \text{cof.} b \text{cof.} c)}; \\
E = \text{fin.} c
\]
\[ \sin C = \frac{\sqrt{(1 - \text{c}a^2 - \text{c}b^2 - \text{c}c^2 + 2 \text{c}a \text{c}b \text{c}c)} - \text{c}f \text{n} \text{i} - \text{c}f \text{n} \text{i} \text{a} b}{\text{f} \text{i} n \text{n} a \text{f} \text{i} n \text{i} b} \]

\[ \text{vbi, loco radicalis ponamus} \]
\[ \sqrt{(x - \text{c}a^2 - \text{c}b^2 - \text{c}c^2 + 2 \text{c}a \text{c}b \text{c}c)} = k \]

\[ \text{et ad calculum contrahendum pro numeratiribus statuamus} \]
\[ \text{c}f \text{a} = a, \text{c}f \text{b} = \beta \text{et} \text{c}f \text{c} = \gamma, \]

\[ \text{vt fit} \]
\[ k = x - \alpha \alpha - \beta \beta - \gamma \gamma + 2 \alpha \beta \gamma. \]

\[ \text{Hoc facto erit} \]
\[ \text{c}f \text{a} = \frac{x - \beta \gamma}{\text{f} \text{i} n \text{n} a \text{f} \text{i} n \text{i} a \text{f} \text{i} n \text{i} c}, \text{c}f \text{b} = \frac{\beta - \alpha \gamma}{\text{f} \text{i} n \text{n} a \text{f} \text{i} n \text{i} a \text{f} \text{i} n \text{i} c}, \text{c}f \text{c} = \frac{\gamma - \alpha \beta}{\text{f} \text{i} n \text{n} a \text{f} \text{i} n \text{i} a \text{f} \text{i} n \text{i} b} \]

\[ \text{§ 8. Coniungamus nunc primo angulos A et B \}
\[ \text{ac reperiemus} \]
\[ \text{f} \text{i} n \text{n} (A \pm B) = \text{f} \text{i} n \text{n} a \text{c}f \text{a} \text{c}f \text{b} + \text{f} \text{i} n \text{n} a \text{c}f \text{a} \text{c}f \text{c} \text{f} \text{i} n \text{n} a \text{c}f \text{b} \text{f} \text{i} n \text{n} a \text{c}f \text{c} \]

\[ \text{Quod si nunc tertium angulum C coniungamus, erit} \]
\[ \text{f} \text{i} n n(A + B + C) = \text{f} \text{i} n a \text{c}f \text{a} \text{c}f \text{b} \text{c}f \text{a} \text{c}f \text{c} + \text{f} \text{i} n a \text{f} \text{i} n a \text{c}f \text{a} \text{c}f \text{c} \text{c} \text{f} \text{i} n a \text{c}f \text{b} \text{c}f \text{a} \text{c}f \text{c} \]

\[ \text{Tantum igitur supereft, vt in his formulis loco literarum} \]
\[ \text{maiusrualorum A, B, C, valores modo assignati substituantur.} \]

\[ \text{Prima Investigatio,} \]
\[ \text{pro fin. S.} \]

\[ \text{§ 9. Cum fit fin. S = fin. (A + B + C), erit} \]
quae expressio cum conflitet quaotr membris, singula seor-
fin empliemus. Erit igitur:

I. fin. A fin. B fin. C = \frac{k}{jm, c^2jm, b^2jm, jm, c^2} \frac{1}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2}.

II. fin. A cof. B cof. C = \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(\beta - a \gamma)(1 - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k{1 - \beta \gamma}}{jm, a^2jm, b^2jm, jm, c^2}.

III. fin. B cof. A cof. C = \frac{k{1 - \beta \gamma}}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(\beta - a \gamma)(1 - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2}.

IV. fin. C cof. A cof. B = \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(\beta - a \gamma)(1 - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2}.

Quia ergo vbique idem habetur denominator
fin. a^2 fin. b^2 fin. c^2 = (1 - a \alpha)(1 - \beta \beta)(1 - \gamma \gamma),
tria membra posteriorea, in vnam summam collecta, dabunt
\frac{k{1 - \beta \gamma}}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(\beta - a \gamma)(1 - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2}.

\text{§ 10. Ad has formulis tractabiliorem reddendas ponamus breuitatis gratia:}

\begin{align*}
\alpha + \beta + \gamma &= p; \\
\alpha \beta + \alpha \gamma + \beta \gamma &= q \text{ et } \alpha \beta \gamma &= r.
\end{align*}

hicque erit

\begin{align*}
\alpha a + \beta \beta + \gamma \gamma &= pp - 2q, \\
nunde fit \\
k k &= \frac{x - p}{p} + 2q + 2r.
\end{align*}

Deinde cum fit

\begin{align*}
pq &= \alpha a \beta + \alpha a \gamma + \beta \beta + \alpha \beta \gamma + \gamma \gamma + \gamma \gamma a + \gamma \gamma \beta + 3a \beta \gamma, \\
\text{erit} \\
\alpha a (\beta + \gamma) + \beta \beta (a + \gamma) + \gamma \gamma (a + \beta) &= pq - 3r.
\end{align*}

quibús valoribus substitutiis terna posteriora membra in-
dièm praebent \frac{k{1 - \beta \gamma}}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(\beta - a \gamma)(1 - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2}.


\text{Dabunt in membro} = \frac{k{1 - \beta \gamma}}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(\beta - a \gamma)(1 - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2}.

\text{Subtracta, relinquit id quo\textit{ quae rimus}, scilicet:}

\begin{align*}
\text{fin. } S &= \frac{k{1 - \beta \gamma}}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(-a + \beta - \gamma + \alpha - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2} \frac{k(\beta - a \gamma)(1 - \beta \gamma)}{jm, a^2jm, b^2jm, jm, c^2}.
\end{align*}

\text{vbi}
vbi observasse iuxtabit, quia, posito $\alpha = x$, denominator 
Euaneicit, eodem casu quoque numeratorem Euaneicere 
debe, quod idem quoque euenire debet casibus $\beta = x$
et $\gamma = x$, ita vt numeratorem necessario habeat factores 
$1 - \alpha; 1 - \beta; 1 - \gamma$, quorum productum cum fit $x - p + q - r$,
per hoc simul numeratorem erit dividibilis, et dividende facta 
quotus repetitur $x + p$; denominator vero, per cundem 
dividendum dividus, fit

$$(x + a)(x + \beta)(x + \gamma) = x + p + q + r,$$

ificque resultat ista formula:

$$\text{fin. } S = \frac{k(1 + p)}{1 + p + q + r},$$

siue valoribus restituitis

$$\text{fin. } S = \frac{(1 + a + \beta + \gamma)(1 + a - \beta - \gamma + a\beta\gamma)}{(1 + a)(1 + \beta)(1 + \gamma)},$$

vbi denotat $a$, cof. $a$; $\beta$, cof. $b$; $\gamma$, cof. $c$. Hancque for-

mulae operae pretium erit aliquot exemplis illuttrare.

§. ix. Exemplum primum. Sint latera $b$ et $c$
quadrantes, ita vt fit $\beta = 0$ et $\gamma = 0$, critique fin. $S = \sqrt{(1 - a\alpha)}$,
ideoque fin. $S = \text{fin. } a$, consequenter ipsa area $S = a$.
Quando autem ambo latera $AB$ et $AC$ sint quadrantes 
et latus $BC = a$, tum ambo anguli $B$ et $C$ erunt recti,
et ob cof. $A = \alpha = \text{cof. } a$, erit angulus $A = \tilde{a}$, hinc-
que summa omnium angulorum $= 180^\circ + a$, ideoque area quae
ta $S = a$.

§. xii. Exemplum secundum. Sit triangulum sphae-
ricum $AB'C$ ad $A$ rectangulum, et cum ex sphareicis 
fit cof. $BC = \text{cof. } A B$ cof. $A C$, erit cof. $\alpha = \text{cof. } b$ cof. $c$,
ideoque $a = \beta \gamma$; quo valore substituto prohibit:

$$\text{fin. } S$$
\[ \begin{align*}
\text{fin. } S &= \frac{(1+\beta+\gamma+\beta\gamma)\sqrt{(1-\beta\beta-\gamma\gamma+\beta\gamma\gamma)}}{(1+\beta)(1+\gamma)(1+\beta\gamma)} \\
\text{Cum igitur fit } V(x - \beta\beta) &= \text{fin. } b, \text{ et } V(x - \gamma\gamma) = \text{fin. } c, \\
\text{erit pro area trianguli rectanguli} & \\
\text{fin. } S &= \frac{\text{fin. } b \text{ fin. } c}{1 + \text{col. } b \text{ col. } c} = \frac{\text{fin. } b \text{ fin. } c}{1 + \text{col. } a}.
\end{align*} \]

Heim § 13. Exemplum tertium. Si triangulum fuerit aequilaterum, seu \( a = \beta = \gamma \), eius area ita expressetur \( \sqrt{ } \) fit fin. \( S = \frac{(1+\beta)(1-\beta\beta)}{(1+\beta)^2} \), bvi formula radicalis factores haber \((x - \alpha)^2(x - 2\alpha)\), vnde ergo fiet fin. \( S = \frac{(1+\alpha)(1-\alpha)}{(1+\alpha)^2} \).

Hinc si terna latera fuerint quadrantes, ideoque \( a = 0 \), erit fin. \( S = 1 \), ideoque \( S = \frac{1}{3} \).

§ 14. Exemplum quartum. Sint omnia latera trianguli, \( a, b, c \) quam minima, quo cavo triangulum sphæricum abit in triangulum planum, et cum fit \( a = \text{cof. } a = 1 - \frac{1}{2} a a + \frac{1}{3} a^3 \) etc., fimilique modo \( \beta = x - \frac{1}{2} b b + \frac{1}{3} b^3 \) etc. et \( \gamma = x - \frac{1}{2} c c + \frac{1}{3} c^3 \) etc., factor rationalis nostrae formulae fiet \( \frac{1}{2} \beta \), neglectis scilicet partibus minimis. At in formula irrationali non solum partes finitae se mutuo defrunt, sed etiam termini, bvi \( a, b, c \) habent duas dimensiones; quamobrem singulas partes vsque ad quattuor dimensiones evolui operat. Habeimus ergo vt sequitur:

\[ \begin{align*}
\alpha a &= x - aa + \frac{1}{2} a^2, \quad \alpha \beta &= x - \frac{1}{2} aa - \frac{1}{3} bb + \frac{1}{2} b^2 + 2abb, \text{ideaque} \\
\beta \beta &= x - bb + \frac{1}{2} b^3, \quad \beta \gamma &= x - \frac{1}{3} aa - \frac{1}{2} bb - \frac{1}{2} c c + \frac{1}{3} a c + \frac{1}{2} b^c, \text{ideaque} \\
\gamma \gamma &= x - cc + \frac{1}{2} c^3 \\
&= aiabb + bacc + \frac{1}{2} bcc. \\
\text{Hinc}
\end{align*} \]
Hinc igitur coquitas polit signum radicale vt sequitur

\[-2 + 2 + a^2 - b^2 + c^2 = \frac{1}{2} a^2 - \frac{1}{2} b^2 - \frac{1}{2} c^2\]

\[+2 - a^2 - b^2 - c^2 + \frac{1}{3} a^3 + \frac{1}{3} b^3 + \frac{1}{3} c^3\]

\[+\frac{1}{2} a^2 b + \frac{1}{2} a c c + \frac{1}{2} b c c,\]

quae, deletis terminis se desinueruntibus, reductur ad hanc:

\[\frac{1}{2} a^2 b + \frac{1}{2} a c c + \frac{1}{2} b c c - \frac{1}{3} a^3 - \frac{1}{3} b^3 - \frac{1}{3} c^3.\]

Quare cum eam area S fit quam minima, idquaque
fin. S = S, habemus aream quaedam:

\[S = \frac{1}{2} \sqrt{\left( a^2 b + \frac{1}{2} a c c + \frac{1}{2} b c c - \frac{1}{3} a^3 - \frac{1}{3} b^3 - \frac{1}{3} c^3 \right)},\]

fiue:

\[S = \frac{1}{2} \sqrt{2 a^2 b + 2 a c c + 2 b c c - a^3 - b^3 - c^3},\]

quae est formula notissima pro area trianguli plani.

**Investigatio secunda,**

pro costitu S.

§. 15. Cum fit cox. S = - cox. (A + B + C), et

qua quatro membra seorsim evoluta dabunt:

I. cox. A fin. B fin. C = \(\frac{h k (\alpha - \beta \gamma)}{\sin \alpha \sin \beta \sin \gamma}\);

II. cox. B fin. A fin. C = \(\frac{k \beta (\alpha - \beta \gamma)}{\sin \alpha \sin \beta \sin \gamma}\);

III. cox. C fin. A fin. B = \(\frac{k \gamma (\alpha - \beta \gamma)}{\sin \alpha \sin \beta \sin \gamma}\);

Pro termino postremo etr primo

cox. A cox. B = \(\frac{(\alpha - \beta \gamma)(\beta - \gamma \gamma)}{\sin \alpha \sin \beta \sin \gamma}\)

hincque

cox. A cox. B = \(\frac{\alpha \gamma - \alpha \beta - \alpha \gamma \gamma - \beta \gamma \gamma + \alpha \beta \gamma + \alpha \gamma \gamma + \beta \gamma \gamma}{\sin \alpha \sin \beta \sin \gamma}\),

\[\text{fin. a fin. b fin. c}\]

§. 16.
§. 16. Quod si iam iterum ponamus $x + y = p$; $a + \gamma = q$ et $\beta + \gamma = r$, tria membra prora, in unam summatum collecta, dabitur: $\frac{kh(p - q)}{jm \alpha \beta \gamma} = q$ quinimum autem membro, si hoc modo repraesentetur:

$$a \beta + \alpha \gamma + a \beta \gamma = q$$

ab: $a \alpha + \beta \beta + \gamma \gamma = p$ $p = 2 q$ et

$$a \alpha \beta \beta + a \alpha \beta \gamma + \beta \beta \gamma \gamma = q - 2 p r$$

Unde hanc formam:

$$\frac{x - p + a r + b s + c r - d r - e r}{jm \alpha \beta \gamma}$$

Quare cum sit $k \kappa = x - p + a r + b s + c r - d r - e r$, omnibus membris collectis hæc hæbimus:

$$\text{cof. } S = \frac{p - q + r - s + t}{jm \alpha \beta \gamma}$$

Quae formula evoluita sit:

$$\text{cof. } S = \frac{x - p + a r + b s + c r - d r - e r}{jm \alpha \beta \gamma}$$

§. 17. Quia hic iterum denominator evanesceit, et quius $a = 1$, $\beta = 1$ et $\gamma = 1$, necesse est ut in idem casibus etiam numerator evanesceat, idemque istum factorem habeat:

$$(x - a)(x - \beta)(x - \gamma) = x - p + q + r$$

Facta igitur hac divisione pro numeratore nancissimur hunc quotum: $p - q - r + p \beta$; pro denominatore autem quotus est:

$$(1 - a)(1 + \beta)(1 + \gamma) = x + \beta + q + r$$

Sicque nactus sumus illam expressionem:

$$\text{cof. } S = \frac{x + p - q + r}{x + p + q + r};$$

ac pro literis $p$, $q$ et $r$ resitutis valoribus, erit.

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\[ \text{cof. } S = \frac{(\alpha + \beta + \gamma)(1 + \alpha + \beta + \gamma) - \alpha \beta + \alpha \gamma - \beta \gamma - \alpha \beta \gamma}{(1 + \alpha)(1 + \beta)(1 + \gamma)} \]

\[ \text{siue etiam} \]
\[ \text{cof. } S = \frac{\alpha + \beta + \gamma + \alpha \beta + \alpha \gamma + \beta \gamma - \alpha \beta \gamma}{(1 + \alpha)(1 + \beta)(1 + \gamma)} \]

\section{§ 18. Exemplum primum.} Sint duo latera \( b \) et \( c \) quadrantes, ideoque \( \beta = 0 \) et \( \gamma = 0 \), quo ergo cafu probit: \( \text{cof. } S = \frac{\alpha(1 + \alpha)}{1 + \alpha} = \alpha = \text{cof. } a \); consequenter erit iterum \( vt \) supra \( S = a \).

\section{§ 19. Exemplum secundum.} Sit triangulum sphaericum rectangulum, existente angulo \( A \) recto, eriique, \( vt \) supra vidimus, \( \text{cof. } a = \text{cof. } b \) \( \text{cof. } c \), siue \( \alpha = \beta \gamma \), quo valore substituto reperitur:
\[ \text{cof. } S = \frac{\beta + \gamma + \alpha \beta + \alpha \gamma + \beta \gamma + \alpha \beta \gamma}{(1 + \beta)(1 + \gamma)(1 + \beta \gamma)} \], siue
\[ \text{cof. } S = \frac{\beta + \gamma(1 + \beta + \alpha \gamma)}{(1 + \beta)(1 + \gamma)(1 + \beta \gamma)} = \frac{\beta + \gamma}{1 + \beta \gamma} \]
Pro eodem vero cafu supra inuenimus \( S = \frac{\alpha(1 - \beta \gamma)}{1 + \beta \gamma} \),
quod egregie congruit, cum hinc fiat
\[ \text{fin. } S^2 = \text{cof. } S^2 = \frac{1 + \alpha \beta \gamma + \beta \gamma}{(1 + \beta \gamma)} \]

\section{§ 20. Exemplum tertium.} Sit triangulum aequilaterum, siue \( \alpha = \beta = \gamma \), erique \( \text{cof. } S = \frac{\alpha + \beta + \gamma}{(1 + \alpha)^2} \).
Supra autem inuenimus \( vt \) hoc cafu
\[ \text{fin. } S = \frac{(1 + \alpha \beta)(1 - \alpha)}{1 + \alpha^2} \]
ad quarum expressionum contentum ostillendum summus triusque formulæ quadratum, ac probitt:
\[ \text{cof. } S^2 = \frac{\alpha \beta + \alpha \gamma + \beta \gamma - 2 \alpha \beta \gamma}{(1 + \alpha^2)} \]
\[ \text{fin. } S^2 = \frac{(1 + \alpha^2 + \beta \alpha \gamma)(1 - \alpha \beta + \alpha \gamma)}{(1 + \alpha^2)} = \frac{1 + \alpha + \beta \alpha - 2 \alpha \beta - 3 \alpha \beta + 2 \alpha^2}{(1 + \alpha)^2} \quad \text{quae} \]
quorum fractionum summa praebet

\[ \frac{x + 6a + 12a^2 + 15a^3 + 15a^4 + 6a^5 + a^6}{1 + \sqrt{a}} - 1. \]

§ 21. Exemplum quartum. Sint latera trianguli
quam minima, et quia etiam area quasi sit evanescentis,
erit \( S = x - \frac{1}{2} SS \); hinc ex formula, per literas \( p, q, r \)
expressa, erit \( x - \frac{1}{2} SS = \frac{p}{r + p} \frac{q}{q + r} \), unde colligitur.

\[ SS = \frac{r + q + s}{s + p + q + r}, \]
et restitutis pro \( p, q, r \) valoribus sit

\[ SS = \frac{k}{(s + a)(s + \beta)(s + \gamma)}; \]

ubi in denominator pro literis \( \alpha, \beta, \gamma \) sufficit scribere
unitatem, quo facto denominator erit \( s \). Supra vero vi-
dimus, pro numeratorem fieri \( k = \sqrt{(x - \alpha \beta - \gamma \gamma + 2 \alpha \beta \gamma)} \)

\[ = \sqrt{(a + b + c + a b c + b b c - a^2 - \gamma^2 - \beta^2 + \alpha^2),} \]
quo valore posito reperitur:

\[ SS = \frac{a a b b + 2 a c c + b b c c - a^2 - b^2 - c^2}{16}, \]

unde sit vtique

\[ S = \frac{1}{2} \sqrt{(2 a a b b + 2 a c c + 2 b b c c - a^2 - b^2 - c^2).} \]

Tertia inuestigatio,
pro tang. \( S \) et tang. \( \frac{1}{2} S \).

§ 22. Postquam pro area nostrorum trianguli sphae-
rici tam fin. \( S \) quam cof. \( S \) inuenimus, sponte fe prodict
loopings iussius areae, sic licet:

\[ \text{tang. } \frac{x + 6a + 12a^2 + 15a^3 + 15a^4 + 6a^5 + a^6}{1 + \sqrt{a}} - 1 \]

quam formulam succinctius in genere exprimere non licet.
\[ \text{§. 23. Verum tangens dimidiae areae, siue tang,}\]
\[ i \cdot S = \frac{\text{fin. } S}{i + \text{cof. } S} \]
retineamus initio literas \( p, q \) et \( r \), ita vt pro numeratore habeamus
\[ \text{fin. } S = \frac{(r + p)q - (r + q + r)}{(r + q + r)}; \]
at vero pro denominatore, ob
\[ \text{cof. } S = \frac{(i + p)(i + q - r)}{i + p + q + r}, \]
erit
\[ 1 + \text{cof. } S = \frac{(i + p)r}{i + p + q + r} \]
quare hie valoribus substitutis reperitur
\[ i + S = \frac{\sqrt{(1 - \alpha \beta \gamma)}}{i + p} \]
et reesstitutis valoribus,
\[ i + S = \frac{\sqrt{(1 - \alpha \beta \gamma)}}{i + p} \]
quae formula ad vsim vsique est aptissima.

\[ \text{§. 24. Exemplum primum. Si bina latera } b \text{ et } c\]
fuerint quadrantes, ideoque \( \beta = \circ \) et \( \gamma = \circ \), erit
\[ i + S = \frac{\sqrt{(1 - \alpha \beta \gamma)}}{i + p} = \frac{\text{fin. } S}{i + \text{cof. } S} \]
unde manifestum est fore tang. \[ i \cdot S = \text{tang. } i + S = \text{ideoque } S = a, \]
vti iam supra inuenimus.

\[ \text{§. 25. Exemplum secundum. Sit triangulum sphæricum ad } A \text{ rectangulum, ideoque } \text{cof. } a = \text{cof. } b \text{ cof. } c\]
et \( \alpha = \beta \gamma \); hoc autem valore substituto reperitur
\[ i + S = \frac{\sqrt{(1 - \beta \gamma \gamma + \beta \gamma \gamma)}}{i + \beta + \gamma + \beta \gamma}; \]
quae fractio, supra et infra diiindendo per \[ \sqrt{(1 + \beta)(1 + \gamma)} \]
reducitur ad hanc:
\[ \text{tang.} \]
tang. \( \frac{1}{2} S = \sqrt{\left(\frac{1 - \beta}{1 + \beta}\right)\left(\frac{1 - \gamma}{1 + \gamma}\right)} \) \\

Erat vero: \\
\( \sqrt{\frac{r - \beta}{r + \beta}} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \beta}} = \text{tang.} \ \frac{1}{2} b \), \\

similique modo \( \sqrt{\frac{r - \gamma}{r + \gamma}} = \text{tang.} \ \frac{1}{2} c \); quocircum resultat sequens 

formula maxime memorabilis: \\
\( \text{tang.} \ \frac{1}{2} S = \text{tang.} \ \frac{1}{2} b \cdot \text{tang.} \ \frac{1}{2} c \),

cuius consensus cum supra inuentis haud difficileter ostenditur.

§ 26. Exemplum tertium. Si triangulum fuerit 
aequilaterum, fit \( a = \beta = \gamma \), erit 
\( \text{tang.} \ \frac{1}{2} S = 2\sqrt{\frac{1 - \alpha}{1 + \alpha}} = \frac{(1 - \alpha)(1 + \alpha)}{1 + \alpha} \); 

vnde catus, quod singula latere sunt quadrantes, idemque 
\( a = \alpha \), erit \( \text{tang.} \ \frac{1}{2} S = 1 \), idemque \( \frac{1}{2} S = 45^\circ \) et \( S = \pi \).

§ 27. Exemplum quartum. Sint denique tria latera \( a, b, c \) quam minimas et quia 
\( \text{tang.} \ \frac{1}{2} S = \frac{1}{2} S \), erit \( S = 2\sqrt{\frac{1 - \alpha^2 - \beta^2}{1 + \alpha + \beta + \gamma}} \).

Nunc igitur pro denominator sufficit summ\( a = \alpha, \beta = \beta, \gamma = \gamma \), ita ut Coefficiens formulæ radicale sit \( \alpha \); 
ipsum autem formulam radicalem iam supra aliquoties vidi-
mus esse 
\( \sqrt{\frac{a b + b c + c a}{a + b + c} - \frac{1}{2} a^2 - \frac{1}{2} b^2 - \frac{1}{2} c^2} \),

vnde area prorsus ut ante exprimitur.

Problema.

§ 28. Proposito angulo solidi \( \angle OBC \); ex tribus Tab. II. 
angulis planis \( BOC = a, \ AOC = b \) et \( AOB = c \) forma- Fig. x6. 
tum, eius veram mensuram assignare.

Solutio
Solutio.

Quoniam huius anguli solidi mensura statui potest acqualis areae trianguli sphærici, cuius latera sint a, b, c, radio spaææ æxistenti = x, ex praecedentibus intelligitur, angulos solidos, perinde ac planos, sue per gradus et minutæ, sue per arcus circulares exprimi poffe. Ponamus igitur S exprimi mensuram anguli solidi propositi, ac positio breuitatis gratia.

cof. a = a, cof. b = β, cof. c = γ,

triplici modo ista mensura S assignari poterit; primo enim eit per sinus:

\[ \sin S = \frac{1 + a + β + γ}{(1 + a)(1 + β)(1 + γ)} \sqrt{(1 - αα - ββ - γγ + 2αβγ)}; \]
deinde per cosinus:

\[ \cos S = \frac{α + β + γ + αα + ββ + γγ + αβ + αγ + βγ - 2αβγ}{(1 + α)(1 + β)(1 + γ)}; \]

tertio vero commodissime per tangentem semissis:

\[ \tan \frac{1}{2} S = \frac{1 - α α - β β - γ γ + 2αβγ}{1 + α + β + γ}. \]

Vbi imprimis notasse iuablit, si omnes tres anguli a, b, c fuerint recti, tum mensura anguli solidi prodire = 90°; id quod mirifice convenit cum communi loquendi more, dum huiusmodi anguli solidi etiam ab opificibus anguli recti vocari solent; ex quo simul intelligere licet, quinam anguli sue maiores sunt minores angulo recto sint reputandi.

Scholion I.

§ 29. Egregium foret, si etsi angulorum solidorum mensura etiam ad eiusmodi eximias proprietates perducet, quales pro figuris planis locum habent; veluti: quod summa angulorum planorum acqualis est duobus rectis.
ätis. Interim tamen talis proprietas in figuris solidis neuntiquam occurrit, ratione nostra mensurae. Neque enim in omnibus Tetraëdis, quae quatuor constant angulis solidis, summa omnium angulorum solidorum eandem quantitatem constituit, sed prout Tetraëdra magis minusque obliqua construuntur, summa quatuor angulorum solidorum modo major modo minor fieri potest. Si enim Tetraëdron regulare examini subiiciamus, cuius singuli anguli solidi ex ternis angulis planis sexaginta graduum formantur, habebimus \( a = \beta = \gamma = \frac{\pi}{3} \); unde cuiusque anguli solidi mensura ita reperitur, vt sit \( \tan \frac{\pi}{3} S = \frac{\sqrt{3}}{2} \), unde ex tabulis colligitur

\[ \frac{1}{2} S = 15^{\circ}. 48' \] sive \( S = 31^{\circ}. 36' \),

ideoque summa omnium quatuor angulorum huius Tetraëdri erit \( 126^{\circ}. 24' \). Nunc consideremus Pyramidend triangularem, cuius basi itidem sit triangulum aequilaterum, vertex autem definit in cuspide acutissimam, cuius itaque mensura euaneat; pro ternis autem angulis solidi ad basin unus angulus erit \( a = 60^{\circ} \), bini vero reliqui \( b = c = 90^{\circ} \), ita vt sit

\[ a = \frac{1}{3} \] et \( \beta = \gamma = 0 \); unde prodit

\( \tan \frac{\pi}{3} S = \frac{\sqrt{3}}{2} \), ita vt sit \( S = 60^{\circ} \);

unde huius Pyramidis summa omnium angulorum solidorum erit \( 180^{\circ} \), cum ante pro Tetraëdro suffisset tantum \( 126^{\circ} \). Quanquam autem in summa angulorum solidorum cuiusque solidi nulla insignis proprietas elucet, in aliis fortasse relationibus ita mensura proprietates haud contemnendas patefacere poterit.

**Scholion II.**

§ 30. Quae habentur sunt tradita ad mensuram eorum angulorum solidorum spectant, qui ex tribus tantum
tum angulis planis sunt compositi. At si angulus solidus ex quattuor pluribusque angulis planis facrit formatus, eius mensura erit area quadrilateri sphærici, vel polygoni plurium laterum, cuius singula latera aequantur angulis planis solidum confinientibus. Tum igitur nihil aliud opus est, nisi ut tale Polygonum in triangula sphærica resolvesetur, et singulorum areae inordinentur, quippe quorum samma dabit mensuram anguli solidi. His autem casibus non sufficir singulos angulos planos tantum nofse, sed in super necessitatem, ut inclinatio mutua binorum pluriumue fit coguita. Haec cum latissim manifesta, hic tantum adiungam dimensionem angulorum solidorum regulariam, qui ex quotcunque angulis, planis inter se aequalibus et pariter inclinatis, formentur.

Problema.

§. 37. Si angulus solidus componatur ex n angulis planis inter se aequalibus; qui singuli sint == a, et aequaliter inter se inclinentur, inuenire mensuram huius anguli solidi.

Solutio.

Si huic angulo solido sphæra concipiatur circumscripta, cuius radius == r, eius mensura erit Polygonum regulare sphæricum, cuinis omnia latera crunt == a, corumque numeros == n; et quia etiam omnes anguli inter se erunt aequales, Polygonurn erit regulare, idque in eius Tab. II. medio dabitur eius centrum, quod sit in $O$; vnum vero Fig. 17. quodque latus Poligoni sit latus $AB==a$, ex cuius terminis ad $O$ ducantur arcus $AO$ et $BO$, qui erunt inter se aequales, ut habeatur triangulum $AOB$. Quia igitur nume-
numerus talium triangularium est \( \equiv n \), erit

\[ \text{angulus } \angle AOB = \frac{2\pi}{n} \]
at si area totius Polygoni fiatutur \( \equiv S \), quae simul erit mensura anguli propositi, area istius trianguli \( \triangle AOB \) erit \( \frac{S}{n} \). iam ex \( O \) in latus \( AB \) ducatur normalis \( OP \), latus \( AB \) hincceans, eritque \( AP = \frac{1}{2} a \), et

\[ \text{angulus } \angle AOB = \frac{\pi}{n} \].

Vocetur iam angulus \( \angle OAB = \phi \), eritque ex Sphaericis

\[ \sin \phi = \frac{\cot \frac{\pi}{n}}{\cot \frac{1}{2} a} \]

Quia igitur hic angulo \( \phi \) etiam acqualis est angulus \( \angle OBA \), summa angulorum trianguli \( \triangle AOB \) erit \( = 2 \phi + \frac{2\pi}{n} \), vnde ablatis duobus rectis obtinebitur area trianguli \( \triangle AOB \)

\[ \frac{S}{n} = 2 \phi + \frac{2\pi}{n} - \pi \],
hincque area totius Polygoni

\[ S = 2n \phi + 2\pi - n\pi = 2n \phi - (n - 2)\pi \],
quae ergo erit mensura anguli solidi regularis propositi.

**Corollarium I.**

§ 32. Si igitur angulus solidus constet ex tribus angulis planis aequalibus \( = a \), ob \( n = 3 \), erit

\[ \sin \phi = \frac{\cot \frac{6\phi}{a}}{\cot \frac{1}{2} a} \]
quo angulo invento erit mensura anguli solidi

\[ S = 6\phi - \pi = 6\phi - 180^\circ \].

**Corollarium II.**

§ 33. Si angulus solidus ex quatuor constet angulis planis inter se aequalibus \( = a \), ob \( n = 4 \) quiatur

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G angu-
angulus $\Phi$, vt sit \(\sin \Phi = \frac{\text{cof.} 45^\circ}{\text{cof.} \frac{1}{2} a}\); atque hinc reperies: \(\tau\)ur mensura anguli solidi \(S = 8 \Phi - 2 \pi = 8 \Phi - 360^\circ\).

**Corollarium III.**

§ 34. Si angulus solidus consistet ex quinque angulis planis inter se aequalibus \(= a\), ob \(n = 5\) quaeatur angulus $\Phi$, vt sit \(\sin \Phi = \frac{\text{cof.} 36^\circ}{\text{cof.} \frac{1}{2} a}\); hinc vero mensura huius anguli solidi erit \(S = 10 \Phi - 3 \pi = 10 \Phi - 540^\circ\).

**Corollarium IV.**

§ 35. Si angulus solidus ex sex consistet angulis planis inter se aequalibus \(= a\), ob \(n = 6\) quaeatur angulus $\Phi$, vt sit \(\sin \Phi = \frac{\text{cof.} 30^\circ}{\text{cof.} \frac{1}{2} a}\); tum vero mensura huius anguli solidi erit \(S = 12 \Phi - 4 \pi = 12 \Phi - 720^\circ\).

**Scholion.**

§ 36. Secundum haec praecipua computemus angulos solidos quinque corporum regularium, quo facilius cos cum angulo recto, qui in solidis pariter est 90 gra- duum, comparare valeamus; vbi quidem conueniet angulos solidos minores quam 90° nomine acutorum, qui autem exceedunt 90° nomine obtusorum insigne.

Mensura angulorum solidorum
Tetraedri.

§ 37. Cum hic terni anguli plani 60 graduum concurrent ad angulos solidos constituentes, erit \(\frac{1}{2} a = 30^\circ\), \(a = 3\).
\( n = 3; \) unde secundum corollarium i. calculus per logarithmos ita institutur:

\[
\begin{align*}
&\text{l. cof. } 60^\circ = 9,6989700 \\
&\text{l. cof. } 30^\circ = 9,9375306 \\
&\text{I sin. } \Phi = 9,7614394 \\
&\text{hincque } \Phi = 35^\circ, 15'. 52'' \\
&\text{ergo } 6 \Phi = 211^\circ, 35', 12'' \\
\end{align*}
\]

unde quisque angulus solidus Tetraèdri reperietur

\[ S = 31^\circ, 35', 12'' ; \]

sicque hic angulus vix superat trientem anguli recti.

**Menfura angulorum solidorum**

**Octaèdri.**

§. 38. Cum quilibet angulus componatur ex quaternis angulis planis 60 graduum, erit \( s = 30^\circ, \) et \( n = 4; \) unde secundum praecepta corollarii ii calculus per logarithmos institutur, uti sequitur:

\[
\begin{align*}
&\text{l. cof. } 45^\circ = 9,8494850 \\
&\text{l. cof. } 30^\circ = 9,9375306 \\
&\text{I sin. } \Phi = 9,9119544 \\
\end{align*}
\]

hincque erit \( \Phi = 54^\circ, 44'. 8'' , \) ergo \( 3 \Phi = 437^\circ, 53', 4'' ; \)

unde anguli solidi Octaèdri mensura erit \( S = 77^\circ, 53', 4'' , \)

qui ergo angulus non multum a recto deficit. Ceterum hic angulus \( \Phi \) est complementum præcedentis ad \( 90^\circ. \)
Mensura angulorum solidorum
Icosaëdri.

§. 39. Cum hic angulus solidus ex quinis angulis planis \( a = 60^\circ \) componatur, erit \( \frac{1}{2} a = 30^\circ \) et \( n = 5 \); vnde ex coroll. 3 calculus ita instituit optimum:

\[
\begin{align*}
I \text{ cof. } 30^\circ &= 9,9079576 \\
I \text{ cof. } 30^\circ &= 9,9375306 \\
I \text{ fin. } \Phi &= 9,9704270
\end{align*}
\]

vnde colligitur \( \Phi = 69^\circ, 5^\prime, 41^{	ext{II}} \), ergo \( 10 \Phi = 690^\circ, 56^\prime, 55^{	ext{II}} \); hinc anguli solidi Icosaëdri mensura erit \( S = 150^\circ, 56^\prime, 55^{	ext{II}} \), qui ergo angulus iam valde est obtusus.

Mensura angulorum solidorum
Hexaëdri.

§. 40. Cum hic singuli anguli solidi consteat terminis angulis planis rectis, erit \( a = 90^\circ \), \( \frac{1}{3} a = 45^\circ \), et \( n = 3 \); hinc ex Coroll. 1. calculus ita institutur:

\[
\begin{align*}
I \text{ cof. } 60^\circ &= 9,6989700 \\
I \text{ cof. } 45^\circ &= 9,8494350 \\
I \text{ fin. } \Phi &= 9,8494350
\end{align*}
\]

ideoque fit \( \Phi = 45^\circ \), ergo \( 6 \Phi = 270^\circ \); vnde mensura anguli solidi Hexaëdri erit \( 90^\circ \), scilicet hic angulus ipse est rectus.

Mensura angulorum solidorum
Dodecaëdri.

§. 41. Cum hic quilibet angulus constet ex terminis
nis planis, quorum singuli continent 108°, erit \( \frac{1}{2} a = 54° \), et \( n = 3 \); vnde calculus secundum coroll. 1, ita institui debet:

\[
\begin{align*}
\text{l col. } 60° & = 9,6939700 \\
\text{l col. } 54° & = 9,7692187 \\
\text{l fin. } \Phi & = 9,9297513
\end{align*}
\]

hincque erit ipse angulus

\( \Phi = 58°, 16°, 54° \), ergo \( 6 \Phi = 349°, 41°, 425° \).

Mensura igitur anguli solidi Dodecaëdræ erit \( 169°, 41°, 42° \), sicque hic angulus Dodecaëdri inter omnia corpora regularia est maximus.

**Scholion.**

§ 42. Quodsi angulus solidus formetur ex sex angulis planis \( a = 60° \), vt fit \( \frac{1}{2} a = 30° \) et \( n = 6 \), corpus regulare inde ortum est ipsa sphaera, in cuius superficie omnes anguli solidi in planum sunt depressi, sicque aequalitatem quatuor angulis rectis; id quod etiam calculus secundum Coroll. 4. institutus declarat:

\[
\begin{align*}
\text{l col. } 30° & = 9,9375306 \\
\text{l col. } 30° & = 9,9375306 \\
\text{l fin. } \Phi & = 10,000000
\end{align*}
\]

hincque angulus

\( \Phi = 90° \) et \( 12 \Phi = 1080° \),

vnde fit angulus solidus \( S = 360° \). Idem evemit si angulus solidus ex quatuor planis rectis componatur, vt fit \( \frac{1}{2} a = 45° \) et \( n = 4 \); tum enim erit

\[ G \]
fin. \( \Phi = \frac{\text{cof.} 45^\circ}{\text{cof.} 30^\circ} = \frac{1}{2} \), ideoque \( \Phi = 90^\circ \).

et angulus solidus \( S = (3 - 4) 90 = 360^\circ \). Denique \( \Phi \) angulus solidus consistet ex tribus planis, ita ut sit

\[ a = 120^\circ, \quad \text{rit} \quad a = 60^\circ \quad \text{et} \quad n = 3; \]

unde iterum sit

fin. \( \Phi = \frac{\text{cof.} 45^\circ}{\text{cof.} 30^\circ} = \frac{1}{2} \), ideoque \( \Phi = 90^\circ \),

et angulus solidus \( S = (6 - 2) 90 = 360^\circ \).