DE OSCILLATIONIBVS
MINIMIS PENDULI QUOTCVNQVE POND-
DVSQVILIS ONVSTI.

Auctore
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Problemata.

Si filo tenuissimo situe grauitatis experti quotun-
que ponduscula A, B, C, D im datis a se inui-
cem interiis fuerint alligata, idque ex puncto O
fusamur et vtcunquc ad motum concitatun oscillati-
iones minimas peragat, eius statum et motum ad
quodvis tempus definire.

Solutio.

§. 1. Ex puncto suspensionis O ducatur recta Tab. III.
verticalis OV, et quicunquc motus pendulz primum Fig. 1.
fuerit impressus elapso tempore = t pendulum te-
uant situm in figura expressum OABC etc. et
ex singulis pondusculis ad verticalem OV agantur
normales AP, BQ, CR, DS etc. Iam quia singu-
gula ponduscula dantur, eorum massa se pondera
defignentur litteris A, B, C, D etc. et quia eorum
interiun etiam dantur ponamus distantias

OA = a; AB = b; BC = c; CD = d etc.

Nam 3

Postero.
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Porro pro singulis pondusculis fluctuantur coordinatae

\[ \text{OP} = x; \quad \text{OQ} = x'; \quad \text{OR} = x''; \quad \text{OS} = x''' \text{ etc.} \]
\[ \text{PA} = y; \quad \text{QB} = y'; \quad \text{RC} = y''; \quad \text{SD} = y''' \text{ etc.} \]

Tum vero ductis verticalibus \( Aq, Br \); \( Cs \) etc. vocentur anguli quibus singula interualla a fitu verticali declinant

\[ \text{AOp} = p; \quad \text{BAq} = q; \quad \text{CBr} = r; \quad \text{DCs} = s \text{ etc.} \]

ex quibus illae coordinatae ita determinantur ut fit

\[
\begin{align*}
x &= a \cos p \\
x' &= a \cos p + b \cos q \\
x'' &= a \cos p + b \cos q + c \cos r \\
x''' &= a \cos p + b \cos q + c \cos r + d \cos s \\
\text{etc.}
\end{align*}
\]

\[ \begin{align*}
y &= a \sin p \\
y' &= a \sin p + b \sin q \\
y'' &= a \sin p + b \sin q + c \sin r \\
y''' &= a \sin p + b \sin q + c \sin r + d \sin s \\
\text{etc.}
\end{align*} \]

§. 2. His positis, pro motu determinando vocetur

tenso fili \( OA = P \)

\[ \text{tenso fili } AB = Q \]

\[ \text{tenso fili } BC = R \]

\[ \text{tenso fili } CD = S \]

atque hinc, \( \text{fi tempus } t \) in minutis secundis exprimatur eiusque differentiale \( dt \) pro constante habeatur, altitudo autem ex qua gravia \( \text{vno minuto } \) secundo libere delabuntur notetur littera \( g \), principia mechanicum sequentes suppediantae aequationes

\[
ad \times \frac{d^2 x}{dt^2}
\]
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\[ \frac{ddx}{s g dt^2} = I - \frac{p \cos \phi}{A} + \frac{Q \cos \theta}{a} \]
\[ \frac{d\phi}{s g dt^2} = I - \frac{Q \cos \phi}{B} + \frac{R \cos \phi \phi}{B} \]
\[ \frac{d\theta}{s g dt^2} = I - \frac{R \cos \phi}{C} + \frac{S \cos \phi \phi}{C} \]
\[ \frac{d\phi dt^2}{s g dt} = I - \frac{S \cos \phi \phi}{D} \]

etc.

harum aequationum numerus, quia duplo major est quam numerus pendulorum, sufficit tam ad singulas tensiones P, Q, R, S etc. quam ad angulos p, q, r, s etc. determinandos pro quouis tempore t.

§ 3. Haec ita se habent in genere quantaequeque etiam fuerint oscillationes, quo autem casu vis vitiores proregredi licet, quam ob rem cogimur investigations nostras tantum ad eos casus accommodare, quibus oscillationes sunt quam minimae, vit in problemate enunciatur. Quin igitur hoc casu omnes anguli p, q, r, s esse debent quam minimi, pro eorum cosinibus scribere licebit vnitatem; pro finibus autem ipsos angulos p, q, r, s etc. Hinc igitur singulae abscissae et applicatae fortientur valores

\[ x = a \]
\[ x' = a + b \]
\[ x'' = a + b + c \]
\[ x''' = a + b + c + d \]

\[ y = ap \]
\[ y' = ap + bq \]
\[ y'' = ap + bq + cr \]
\[ y''' = ap + bq + cr + ds \]

etc.

Quia igitur abscissae hoc casu sunt constantes, earum differentialia evanescunt; ex quibus nascentur frequentes aequationes:

\[ o = A \]
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\[ o = A - P + Q; \quad o = B - Q + R; \quad o = C - R + S; \quad o = D - S \]

ex quibus flatim singulae tensiones facillime definimur, sic licet

\[ S = D; \quad R = C + D; \quad Q = B + C + D \text{ et } P = A + B + C + D; \text{ etc.} \]

hinc ad calulum contrahendum ponamus breuitatis gratia

\[
\frac{p}{a} = 1 + \frac{b + c + d}{a} \quad \text{hinc erit} \quad \frac{q}{a} = \frac{b + c + d}{a} - a - b
\]

\[
\frac{q}{b} = 1 + \frac{c + d}{b} = c
\]

\[
\frac{r}{c} = 1 + \frac{d}{c} = r
\]

\[
\frac{s}{d} = 1 = s
\]

§ 4. Quod si iam pro applicatis \( p, q, r, s \) itemque pro tensionibus \( P, Q, R, S \) etc. suos scribamus valores, adipiscemur sequentes aequationes differentiales fecundis graduis:

I. \( \frac{\ddp + \bqq \cdot d}{d^2} = \frac{1}{a - b} (a - b) \)

II. \( \frac{\ddp + \bqq \cdot d}{d^2} = \frac{1}{c - d} (c - d) \)

III. \( \frac{\ddp + \bqq \cdot d}{d^2} = \frac{1}{e - f} (e - f) \)

IV. \( \frac{\ddp + \bqq \cdot d}{d^2} = \frac{1}{g - h} (g - h) \)

Sicque totum negotium \( a, b \) resolutionem harum aequationum differentio-differentialium reducitur, quae utique artificialis proenus singularia posuit.

§ 5. Quia in omnibus his aequationibus variabilis \( p, q, r, s \) etc. tantum unicum tenet dimensionem, euidens est, his aequationibus satissimi posse, \( \text{fi in tes} \)
$\alpha$ inter quantitates $p, q, r, s$ certae rationes conflantes statuuntur. Sit igitur

$p = A z; q = B z; r = C z; s = D z$

sic enim illae aequationes sequentes induent formas:

I. \[ \frac{d^2 z}{az^2} = -\alpha A z + (\alpha - 1) B z \]

II. \[ \frac{\beta d^2 z}{az^2} = -\beta B z + (\beta - 1) C z \]

III. \[ \frac{\gamma d^2 z}{az^2} = -\gamma C z + (\gamma - 1) D z \]

IV. \[ \frac{\delta d^2 z}{az^2} = -\delta D z + \gamma D z \]

quae aequationes cum omnibus inter se compleantur debeat, singulas ad hanc formam reducemus:

\[ \frac{d^2 z}{az^2} = -\frac{\gamma}{k} \]

quo valore in singulis substituto nuncsisemur sequentes quatuor aequationes inter meras quantitates conflantes, sicilicet

I. \[ \frac{d^2 z}{az^2} = -\alpha A z + (\alpha - 1) B z \]

II. \[ \frac{d^2 z}{az^2} = -\beta B z + (\beta - 1) C z \]

III. \[ \frac{d^2 z}{az^2} = -\gamma C z + (\gamma - 1) D z \]

IV. \[ \frac{d^2 z}{az^2} = -\delta D z + \gamma D z \]

§ 6. Ex his iam aequationibus determinare licet coefficientes assumtos $A, B, C, D$ etc. Ex prima enim erit

$B =\frac{\alpha}{a^2} (a^2 - 1);$ ex secunda erit

$C =\frac{\beta}{\gamma} (\beta - 1) (B - \frac{\alpha}{a})$ siue $C = \frac{\beta}{\gamma} (B - \frac{\alpha}{a}) - \frac{\alpha}{\gamma}$

eodem modo ex tertia elicitus

$D = \frac{\delta}{\gamma} (\gamma - 1) (\gamma - \frac{\alpha}{a}) - \frac{\alpha}{\gamma} + \frac{\alpha}{\gamma}$

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Qui valores in quarta substituti producent aequationem algebraicam, ex qua quantitatem incognitam k determinari opportet; ubi aequatio tot inuvult radices, quot dantur ponuntulae ita vt pro k tolicem diversi valores sint prodituri. Quarta autem aequatio quae hic est ultima habe forma repraesentatur:

\[ a + b \cdot c + d \cdot k = 0 \]

§ 7. Substituamus nunc successiue valores ex prioribus aequationibis inventos in posterioribus; et quia erat:

\[ B = \frac{a}{a - k} \left( x - \frac{a}{k} \right) \]

\[ C = \left( \frac{x}{k - (\frac{a}{k})} \right) \left( x - \frac{a}{k} \right) \left( \frac{a}{k} \right) \]

\[ D = \left( \frac{x}{k - (\frac{a}{k})} \right) \left( \frac{a}{k} \right) \left( \frac{a}{k} \right) \left( \frac{x}{k - (\frac{a}{k})} \right) \]

qui values ad sequentes formas reducuntur:

\[ (x - \frac{a}{k}) \]

\[ (x - \frac{a}{k}) \frac{a}{k} = x - \frac{a}{k} \]

\[ (x - \frac{a}{k}) \left( \frac{a}{k} - x \right) = a \cdot \frac{a}{k} - a \left( x + \frac{a}{k} \right) - \frac{a}{k} \]

\[ (x - \frac{a}{k}) \left( \frac{a}{k} - x \right) \left( \frac{a}{k} - x \right) = \frac{a}{k} \cdot \frac{a}{k} - \frac{a}{k} \left( x + \frac{a}{k} \right) - \frac{a}{k} \cdot \frac{a}{k} \]

§ 8. Quod si iam itos valores in aequatione inventa substituamus, pro determinazione quantitatis k prohibirem aequatio quarti gradus ad quam commodius inueniendam illam aequationem multiplicemus per:

\[ (x - \frac{a}{k}) \left( \frac{a}{k} - x \right) \left( \frac{a}{k} - x \right) \]

vt habeatur illa aequatio:

\[ \frac{(x - \frac{a}{k}) \left( \frac{a}{k} - x \right) \left( \frac{a}{k} - x \right) \left( \frac{a}{k} - x \right)}{k} = 0 \]

facto:
facto autem calculo aequatio isla biquadratica ita reperietur expressa:

\[
\begin{align*}
\alpha \beta \gamma \cdot k^2 - \alpha \beta \gamma (a + b + c + d) k + b c a \gamma & + c d a \gamma \\
+ a c \gamma (a + b - 1) & + a d (a \beta + \alpha \gamma + \beta \gamma - a - b - \gamma + 1) \\
- b c d \alpha & - a b c \gamma \\
- a b d (a + \beta - 1) & - a c d (a + b - 1) \\
\end{align*}
\]

vbi observasse inuabit, primo litteras \(a, b, c, d\) semper denotare distantias positivas; sum vero litteras \(\alpha, \beta, \gamma, \delta\), esse numeros positivos atque adeo unitate magnores. Hinc enim ratio intelligi poterit, cur omnes quattuor radices huius aequationis proditurne sint reales; eas autem omnes esse positivas permutato signorum declarat.

§. 9. Ipsi resolutioni huius aequationis hic non immoramus, quando quidem si numerus pondificalorum est maius a tali investigatione propterea abstinere cogeremur: designemus igitur quattuor huius aequationis radices litteris \(k, k', k''\) et \(k'''\) ex quarum singulis peculiares valores pro litteris \(\Lambda, \beta, \Gamma\) et \(\Delta\) coligemus, quos pariter hoc modo designemus \(\Lambda', \Lambda''\), \(\beta', \beta''\), \(\Gamma', \Gamma''\) et \(\Delta', \Delta''\); vbi quidem patet, litteras \(\Lambda, \Lambda', \Lambda'', \Lambda'''\) arbitrio nostro open-
penitus relinqui; ita vt hoc modo quatuor habeamus quantitates pro lubitu accipiendas.

$. 10. Prosequamur igitur nostrum calculum pro sola radice $k$, cui respondent coefficiens $A$, $B$, $C$, $D$, quandoquidem quod pro hac radice fuerit commutum facillime quoque ad reliquas radices applicatur. Cum igitur statuissimus hanc aequationem differentiam secundi gradus \( \frac{d^4 z}{dt^4} = \frac{c}{k} \), ita vt sit \( \frac{d^4 z}{dt^4} + \frac{c}{k} z = 0 \) si ponamus \( \frac{c}{k} = \lambda \), vt sit \( \lambda = \sqrt{\frac{c}{k}} \)
fi quidem $k$ semper est quantitas realis positiva, nostrum est post duplicem integrationem prodire $z = f \sin (\lambda t + \phi)$; ubi $\phi$ est angulus ab arbitrio nostro pendens. altera autem constans arbitraria $f$ sine restrictione unitali aequalis ponendi: propterea quod coefficiens $A$ iam est arbitrarius. Hinc igitur ad quadam vis tempor $t$ singuli anguli $p$, $q$, $r$, $s$ ita determinabuntur, vt sit

I. $p = A \sin(\lambda t + \phi)$; $q = B \sin(\lambda t + \phi)$; $r = C \sin(\lambda t + \phi)$ et $s = D \sin(\lambda t + \phi)$

namille modo si ex reliquis radicibus, $k^I$, $k^II$, $k^III$
ponamus

$\lambda = \sqrt{\frac{c}{k^I}}$, $\lambda^I = \sqrt{\frac{c}{k^II}}$, $\lambda^II = \sqrt{\frac{c}{k^III}}$

praeter illam solutionem adhuc habeimus tres sequentes

II. $p^I = A^I \sin(\lambda^I t + \phi^I)$; $q^I = B^I \sin(\lambda^I t + \phi^I)$; $r^I = C^I \sin(\lambda^I t + \phi^I)$ et $s^I = D^I \sin(\lambda^I t + \phi^I)$

III. $p^I = A^{II} \sin(\lambda^{II} t + \phi^{II})$; $q^I = B^{II} \sin(\lambda^{II} t + \phi^{II})$; $r^I = C^{II} \sin(\lambda^{II} t + \phi^{II})$ et $s^I = D^{II} \sin(\lambda^{II} t + \phi^{II})$

IV.
§. xi. Singulae autem haec quatuor solutiones maxime sunt particulares: propterque quod duas tantum constantes arbitrariae involuant, sicut $A$ et $B$, dum solutio generalis ob quatuor aequationes differentiales octo constantes arbitrariae complecti deberet. Qualis igitur motus singulis respondet operae pretium erit accuratus investigare: ac primo quidem quoniam pro qualibet caso quatuor anguli $p$, $q$, $r$, $s$ eandem perpetuo inter se servant rationem, motus erit maxime regularis et pendulum perinde oscillationes suas pereget ac si effert simplex; atque quia elapso tempore $t = \frac{2\pi}{\lambda} \text{ si loco } t$ scribamus $t = \frac{2\pi}{\lambda}$ singuli anguli in eundem statum revolutatur, ideoque pendulum interea duos oscillationes absolvivit censetur, sicque tempus unius oscillationis erit $= \frac{\pi}{\lambda} = \frac{\pi}{\sqrt{g} \cdot A}$, quot adeo tempus in minutis secundis exprimitur. Eodem modo pro secunda radice $k'$ erit tempus unius cuiusque oscillationis $= \frac{\pi}{\sqrt{g} \cdot A}$ pro tertia radice $= \frac{\pi}{\sqrt{g} \cdot A}$ et pro quarta $= \frac{\pi}{\sqrt{g} \cdot A}$.

§. xii. Cuius igitur haec quatuor solutiones simplex problemati nostro satisfaciant, quoniam in aequationibus differentiis differentialisibus ad quas nos solutio perduxit singulae quantitates $p$, $q$, $r$, $s$ unumque versus tantum dimensionem tenent, solutiones illae particulares quomodocunque inter se combinentur problema.
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blemati pariter satisfacient, unde sequens solutio generalis consistit:

\[ p = A \sin(\lambda t + \varphi) + A' \sin(\lambda' t + \varphi') + A'' \sin(\lambda'' t + \varphi'') + A''' \sin(\lambda''' t + \varphi''') \]

\[ q = B \sin(\lambda t + \varphi) + B' \sin(\lambda' t + \varphi') + B'' \sin(\lambda'' t + \varphi'') + B''' \sin(\lambda''' t + \varphi''') \]

\[ r = C \sin(\lambda t + \varphi) + C' \sin(\lambda' t + \varphi') + C'' \sin(\lambda'' t + \varphi'') + C''' \sin(\lambda''' t + \varphi''') \]

\[ s = D \sin(\lambda t + \varphi) + D' \sin(\lambda' t + \varphi') + D'' \sin(\lambda'' t + \varphi'') + D''' \sin(\lambda''' t + \varphi''') \]

in his enim formulis octo occurrunt constantes arbitrarie, sic cidet quatuor coefficientes \( A, A', A'', A''' \) quippe per quos reliqui determinantur; tum quatuor anguli \( \varphi, \varphi', \varphi'', \varphi''' \), quemadmodum gemina integra quatuor illarum equationum postulat. Hinc igitur pariet, principium illustris D. Bernoulli, quod omnes hujusmodi oscillationes ex unibus vel pluri- bus motibus oscillatoris simplicibus et regularibus componi statuit, omnino in primis notus principiis esse fundatum atque adeo ex iis immediate deduci posse.

§ 13. Ope harum igitur formularum ad quodvis tempus \( t \) singuli illi anguli \( p, q, r \) et \( s \) assignari l acque status penduli definiti poterit. Quin etiam norem angulorum variationes momentaneae celeritates praebebunt, quibus status penduli quouis temporis momento \( dt \) immutatur. Cum enim formulis

\[ \frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt} \text{ et } \frac{ds}{dt} \]

exprimant celeritates angulares, quibus illi anguli temporis \( dt \) augentur, haec celeritates ita se habebunt

\[ \frac{dp}{dt}. \]
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\[
\frac{\dd x}{\dd t} = \lambda \mathcal{A} \cos(\lambda t + \mathcal{D}) + \lambda' \mathcal{A}' \cos(\lambda' t + \mathcal{D}') + \lambda'' \mathcal{A}'' \cos(\lambda'' t + \mathcal{D}'')
\]
- \lambda''' \mathcal{A}''' \cos(\lambda''' t + \mathcal{D}''')

\[
\frac{\dd y}{\dd t} = \lambda \mathcal{B} \cos(\lambda t + \mathcal{D}) + \lambda' \mathcal{B}' \cos(\lambda' t + \mathcal{D}') + \lambda'' \mathcal{B}'' \cos(\lambda'' t + \mathcal{D}'')
\]
- \lambda''' \mathcal{B}''' \cos(\lambda''' t + \mathcal{D}''')

\[
\frac{\dd z}{\dd t} = \lambda \mathcal{C} \cos(\lambda t + \mathcal{D}) + \lambda' \mathcal{C}' \cos(\lambda' t + \mathcal{D}') + \lambda'' \mathcal{C}'' \cos(\lambda'' t + \mathcal{D}'')
\]
- \lambda''' \mathcal{C}''' \cos(\lambda''' t + \mathcal{D}''')

\[
\frac{\dd r}{\dd t} = \lambda \mathcal{D} \cos(\lambda t + \mathcal{D}) + \lambda' \mathcal{D}' \cos(\lambda' t + \mathcal{D}') + \lambda'' \mathcal{D}'' \cos(\lambda'' t + \mathcal{D}'')
\]
- \lambda''' \mathcal{D}''' \cos(\lambda''' t + \mathcal{D}''')

Acque omnia sumus adepti, quae circa solutionem huius problematis desiderati possunt.

Corollarium.

S. 14. Maxima igitur difficultas in resolucionem equationis algebrae: ex qua omnes valores literae \( k \) determinari oportet, occurrit; praecipue si pendulum pluribus ponderis suosierit operatum. Tum vero etiam quamadmodum pro singulis valoribus ipsius \( k \) coeicientes \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) et \( \mathcal{D} \) definiri commodo quant nonnullius factis liquet; pro pluribus quaedam: quae a bunc ponderuis. Quae igitur hanc: investigationem faciorem reddamus, differentias inter binas: equationes fere invenientes (S. 5.) exhibitis consideremus.

I \( \frac{\dd^2 x}{\dd t^2} = -a \mathcal{A} + (x-1) \mathcal{B} + \cos(x-1) \mathcal{D} \)

I-II. \( \frac{\dd^2 y}{\dd t^2} = -a \mathcal{A} + \mathcal{B}(x + \gamma - 1) + \mathcal{C}(\gamma - 1) \)

II-III. \( \frac{\dd^2 z}{\dd t^2} = -\mathcal{A} + \mathcal{B}(x + \gamma - 1) - \mathcal{C}(\gamma - 1) \)

III-IV. \( \frac{\dd^2 r}{\dd t^2} = -\gamma \mathcal{C} + \gamma \mathcal{D} \)
vnde facile patet quomodo haee aequalititates sint continuandae, si pondusculorum numerus fuerit maior.

§. x5. Supra litterae B, C et D ex prima A determinauimus; nunc autem a postrema incipientes singulas ex ultima D deriuemus, vnde fit ut sequitur

\[ \gamma C = D (\gamma - \frac{a}{k}) \]
\[ \xi B = C (\xi + \gamma - \frac{a}{k}) - D (\gamma - \xi) \]
\[ \alpha A = B (\alpha + \xi - \frac{b}{k}) - C (\xi - \alpha) \]

vnde reperimus

\[ \frac{\xi}{D} = \frac{\gamma}{\xi} (\gamma - \frac{a}{k}) \]
\[ \frac{\xi B}{B D} = \frac{1}{\xi \gamma} (\gamma - \frac{a}{k})(\xi + \gamma - \frac{b}{k}) - \frac{\gamma - \xi}{\xi \gamma} \]
\[ \frac{\xi B}{D} = \frac{1}{\alpha \xi \gamma} (\gamma - \frac{a}{k})(\xi + \gamma - \frac{b}{k})(\alpha + \xi - \frac{b}{k}) - \frac{\gamma - \xi}{\alpha \xi \gamma} \]

qui valores in prima aequatione substituti producent illum aequacionem:

\[ \varphi = -\frac{1}{\alpha \xi \gamma} (\alpha - \frac{a}{k})(\alpha + \xi - \frac{b}{k})(\xi + \gamma - \frac{a}{k})(\gamma - \frac{a}{k}) \]
\[ + \frac{\gamma - \xi}{\alpha \xi \gamma} (\alpha - \frac{a}{k})(\alpha + \xi - \frac{b}{k}) + \frac{\gamma - \xi}{\alpha \xi \gamma} (\alpha - \frac{a}{k})(\gamma - \frac{a}{k}) \]
\[ + \frac{\xi - \alpha}{\alpha \xi \gamma} (\xi + \gamma - \frac{a}{k})(\gamma - \frac{a}{k} - \frac{b}{k}) \]

quia aequatio manifesto ascendet ad quantum ordinem, ex qua incognitae \( k \) quoniam valores inueiligari opportet; hocque modo operatio inflitui facile poterit, si pondusculorum numerus fuerit maior.

Scho-
§ 16. Quamvis autem haec solutio sit maximam maxime elegans, et problematis perfectissime satisfaciat, tamen maximae occurunt difficultates, si iam ad cunctum determinatum applicare voluerimus. Quod si enim pro statu initiali vbi \( t = 0 \) singulis angulis \( \alpha, \beta, \gamma \), dato valore tribuere velimus, simulque singulis pondusculis datis celeritates angulares, ad octo aequationes perueniemus, quae similis erunt duabus aequationibus ex angulo \( \gamma \) natis: si enim requiratur vt initio fuerit angulus \( \beta = f \) eisque celeritas angularis \( i \) haec duae obtinatur aequationes:

\[
\begin{align*}
 f &= \alpha \sin \beta + \beta \sin \gamma + \gamma \sin \alpha + \alpha \sin \beta + \beta \sin \gamma + \gamma \sin \alpha + \lambda \alpha \cos \beta + \lambda \beta \cos \gamma + \lambda \gamma \cos \alpha, \\
 i &= \lambda \alpha \cos \beta + \lambda \beta \cos \gamma + \lambda \gamma \cos \alpha
\end{align*}
\]

similesque binae aequationes obtinuuntur pro reliquis pondusculis. Nunc igitur requiritur vt ex his octo aequationibus octo illae constantes arbitrariae
\( \alpha, \beta, \gamma, \lambda \) et \( \alpha, \beta, \gamma, \lambda \)
definiantur, quem sane laborem. Vix quisquam exsequatur: si modo corpusculorum numerus ternarium superauerit; quamobrem iam dudum non dubitati assumuere, solutionem haec quantumuis elegantem et perfectam plane esse ineptam, vt ad cunctum determinatos, quibus penduli status initialis praescribitur ad aptari possit. Ex quo manifesto sequitur, si problema proponatur, vt si penduli datus status et motus initio imprimitur, motus deinceps secutus definiri debet, longe aliam solutionem requiri, quae proinde ab hac maxime discrepare debeat.

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quo omnia ponduscula sunt aequalia eorumque interiulla etiam aequalia = a; ponatur autem breuitatis gratia \( \frac{a}{k} = u \).

§. 17. Sit primo pondusculorum numerus = 2 et \( t = 2 \) et \( \varepsilon = 1 \), unde aequationes ex §. 13. erunt

\[ o = -A(2-u) + B \text{ et } Bu = -2 A + 2 B; \]

ex postiore sequitur

\[ A = \frac{B(2-2u)}{2} \]

qui valor in priore substitutus dat

\[ o = -\frac{B(2-2u)^2}{2} + B \text{ tunc } (2-u)^2 - 2 = 0 \]

unde statim deducitur

\[ 2 - u = \pm \sqrt{2} \] ideoque \( u = 2 \pm \sqrt{2} = \frac{a}{k} \).

tum vero \( A \) per \( B \) ita exprimitur vt sit \( A = \frac{B(2-2u)}{2} \) vbi \( B \) pro labiun accipit potest. Quare si bin 
valores ipsius \( k \) sint \( k \) et \( k' \), hisque respondent litterae \( A \) et \( B \) solution in his aequationibus continetur

\[ \lambda = \sqrt{\frac{a}{k}} \text{ et } \lambda' = \sqrt{\frac{a}{k'}} \]

\[ p = A \sin(\lambda t + \theta) + A' \sin(\lambda' t + \theta') \] et

\[ q = B \sin(\lambda t + \theta) + B' \sin(\lambda' t + \theta'). \]

§. 18. Sit pondusculorum numerus = 3, eritque \( a = 3 \), \( t = 2 \) et \( \varepsilon = 1 \), unde aequationes nostrae erunt

\[ o = -A(3-u) + 2 B \text{ sive } o = \frac{A(3-u)}{2} - 2 B \]

\[ Bu = -3 A + 4 B - C \]

\[ Cu = -2 B + 2 C \]

ex
PENDULI MULTIMEMBRIS.

Ex hac fit \( \mathfrak{A} = \mathfrak{E}(1-u) \) tum vero experiore;
\[ \mathfrak{A} = \mathfrak{E}(\sigma - u) \frac{1 - u}{1} - \mathfrak{E} = \mathfrak{E}(\sigma - \sigma u + u u) \]
vnde aequatio prodit
\[ \frac{(2-u)(1-u)}{1} - \frac{(5-u)}{1} - (2-u) = 0 \]
\[ \sigma - \sigma u - 9 u u - u^2 = 0 \]
cuius ergo dabuntur tres radices, ideoque valores pro
\( k, k', k'' \), ex quibus tota solutio facile conficitur.

§. 19. Sit ponensculorum numerus = 4, erit
\( \alpha = 4, \beta = 3, \gamma = 2, \delta = 1 \), vnde nostrae aequatiores erunt
\[ \frac{2}{1} = \mathfrak{A}(4-u) - 3 \mathfrak{B} \]
\[ \mathfrak{B} u = - 4 \mathfrak{A} + 9 \mathfrak{B} - 2 \mathfrak{C} \]
\[ \mathfrak{C} u = - 3 \mathfrak{B} + 4 \mathfrak{C} - \mathfrak{D} \]
\[ \mathfrak{D} u = - 2 \mathfrak{C} + 2 \mathfrak{D} \]

Ex ultima fit
\[ \mathfrak{C} = \mathfrak{D}(e-u) \]
\[ \mathfrak{B} = \mathfrak{D}(e - u) \frac{1 - u}{1} - \mathfrak{B} = \mathfrak{D} \frac{(6 - 6 u + u u)}{1} \]
\[ \mathfrak{A} = \mathfrak{D} \frac{(24 - 36 u + 12 u u - u^2)}{1} \]
qui valores substituti hanc praebent aequationem
\[ u^2 - 16 u^3 + 72 u u - 96 u + 24 = 0 \]
cuius quattuor radices quaerit oportet.

§. 20. Sit numerus ponensculorum = 5, erit
\( \alpha = 5, \beta = 4, \gamma = 3, \delta = 2, \epsilon = 1 \), et aequatiores nostrae erunt

\[ P \ P \ A \]
\[ \sigma = \sigma \]
DE OSCILLATIONIBUS

\[ \sigma = A (x - u) - 4 B \]
\[ B u = - 5 A + 8 B - 3 C \]
\[ C u = - 4 B + 6 C - 2 D \]
\[ D u = - 3 C + 4 D - E \]
\[ E u = - 2 D + 2 E \]

hic ex tribus posterioribus colligimus:

\[ D = \frac{E}{2} (x - u) \]
\[ C = \frac{E}{6} (6 - 6 u + u u) \]
\[ B = \frac{E}{2} (2x - 36 u + 12 u u - u^2) \]

rum vero inuenitur

\[ A = \frac{E}{100} (120 - 240 u + 120 u u - 20 u^2 + u^3) \]

quibus valoribus substitutis aequatio sequens resultat

\[ u^2 - 25 u + 200 = 600 u u + 600 u - 120 = 0 \]

§. 21. Hinc generaliter si numeros pondusculorum fuerit \( n \), aequatio ex qua \( a \) definiiri debet per legitimam inductionem colligitur fore

\[ o = \frac{x - n u + \frac{n(n - 1)m}{1} u u + \frac{n(n - 1)(n - 2)m}{2} u u u + \frac{n(n - 1)(n - 2)(n - 3)m}{3} u u u u + \frac{n(n - 1)(n - 2)(n - 3)(n - 4)m}{4} u u u u u + \ldots }{1^2, 2^2, 3^2, 4^2, 5^2, \ldots } \]

deinde vero si coëfficientium \( A, B, C, D \) ultimus fuerit \( n \) singuli sequenti modo determinabuntur

\( \sigma = 1 - \frac{n^2}{1^2} u + \frac{1 + n(n - 1)}{1^2} u u u + \frac{1 + n(n - 1)(n - 2)}{1^2} u u u u + \frac{1 + n(n - 1)(n - 2)(n - 3)}{1^2} u u u u u + \ldots \]

etc.
PĘNDULI MULTIMEMBRIS.

Tantum igitur reslat methodus, cuius opus illius aequationis $n$ radices elici quasqu, quippe quibus inuentis solutio completa hujus casus habeitur.

§. 22. Aquatione illa ordinis $n$, ex qua valores litterae $u$ definire oportet etiam hoc modo concinnius referri potest

$$C = u^n - \frac{n^2}{2} u^{n-1} + \frac{n^4(n^2-1)^2}{8} u^{n-2} - \frac{n^6(n-1)^2(n+1)}{8} u^{n-3} + \frac{n^8(n-1)^3(n+1)^3}{32} u^{n-4} \text{ etc.}$$

hinc autem coefficientes $A$, $B$, $C$, $D$ etc. etiam hoc modo exhiberi possunt

$$\pm 1, 2, 3 \ldots n \frac{\delta}{\delta u} = u^{n-1} - \frac{n(n-1)}{2} u^{n-2} + \frac{n(n-1)^2}{2} u^{n-3} - \frac{n(n-1)^3}{6} u^{n-4} \text{ etc.}$$

$$+ 1, 2, 3 \ldots (n-1) \frac{\delta}{\delta u} = u^{n-2} - \frac{n(n-1)(n+1)}{2} u^{n-3} + \frac{(n-1)(n+1)^2}{2} u^{n-4} - \frac{(n-1)^3}{6} u^{n-5} \text{ etc.}$$

$$+ 1, 2, 3 \ldots (n-2) \frac{\delta}{\delta u} = u^{n-3} - \frac{(n-2)(n+1)}{2} u^{n-4} + \frac{(n-2)(n+1)^2}{2} u^{n-5} - \frac{(n-1)^2}{6} u^{n-6} \text{ etc.}$$

$$+ 1, 2, 3 \ldots (n-3) \frac{\delta}{\delta u} = u^{n-4} - \frac{(n-3)(n+1)}{2} u^{n-5} + \frac{(n-3)(n+1)^2}{2} u^{n-6} - \frac{(n-2)^2}{6} u^{n-7} \text{ etc.}$$

etc. etc.

vbi signorum ambiguum superiorum sunt accipienda
et $n$ fuerit numerus impar, inferiora autem $S$ par.