CAPUT IX.

DE EVOLUTIONE INTEGRALIUM PER PRODUCTA INFINITA.

Problema 43.

356.

Valorem hujus integralis $\int \frac{\partial x}{\sqrt{(1-xx)}}$, quem casu $x = 1$ recipit, in productum infinitum evolvere.

Solutio.

Quemadmodum supra formulas altiores ad simplicem reduximus, ita hic formulam $\int \frac{\partial x}{\sqrt{(1-xx)}}$ continuo ad altiores perducamus. Ita cum posito $x = 1$ sit

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}} = \frac{m+1}{m} \int \frac{x^{m+1}\partial x}{\sqrt{(1-xx)}}, \text{ erit}$$

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2}{1} \int \frac{xx\partial x}{\sqrt{(1-xx)}} = \frac{2.\,4}{1.\,3} \int \frac{x^4\partial x}{\sqrt{(1-xx)}}$$

$$= \frac{2.\,4.\,6}{1.\,3.\,5} \int \frac{x^6\partial x}{\sqrt{(1-xx)}} \text{ etc.}$$

unde concludimus fore indefinite:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2.\,4.\,6.\,8 \ldots\ldots 2i}{1.\,3.\,5.\,7 \ldots\ldots (2i-1)} \int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}}$$

atque adeo etiam si pro i sumatur numerus infinitus. Nunc simili modo a formula $\int \frac{x\partial x}{\sqrt{(1-xx)}}$ ascendamus, reperiemusque

$$\int \frac{x \partial x}{\sqrt{(1-xx)}} = \frac{3.5.7.9 \ldots (2i+1)}{2.4.6.8 \ldots 2i} \int \frac{x^{2i+1} \partial x}{\sqrt{(1-xx)}},$$

atque observo, si i sit numerus infinitus, formulas istas

$$\int \frac{x^{2i} \partial x}{\sqrt{(1-xx)}} \text{ et } \int \frac{x^{2i+1} \partial x}{\sqrt{(1-xx)}}$$

rationem aequalitatis esse habituras. Ex reductione enim principali perspicuum est, si m sit numerus infinitus, fore

$$\int \frac{x^{m-1} \partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+1} \partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+3} \partial x}{\sqrt{(1-xx)}}$$

atque adeo in genere $\int \frac{x^{m+\mu} \partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+\nu} \partial x}{\sqrt{(1-xx)}}$ quantumvis magna fuerit differentia inter μ et ν, modo finita. Cum igitur sit $\int \frac{x^{2i} \partial x}{\sqrt{(1-xx)}} = \frac{x^{2i+1} \partial x}{\sqrt{(1-xx)}}$, si ponamus:

$$\frac{2.4.6 \ldots 2i}{1.3.5 \ldots (2i-1)} = \mathrm{M} \text{ et } \frac{3.5.7.9 \ldots (2i+1)}{2.4.6.8 \ldots 2i} = \mathrm{N}, \text{ erit}$$

$$\int \frac{\partial x}{\sqrt{(1-xx)}} : \int \frac{x \partial x}{\sqrt{(1-xx)}} = \mathrm{M} : \mathrm{N} = \frac{\mathrm{M}}{\mathrm{N}} : 1, \text{ posito } x = 1.$$

At est $\int \frac{x \partial x}{\sqrt{(1-xx)}} = 1$ et $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$,

unde colligitur $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\mathrm{M}}{\mathrm{N}}$, quia producta M et N ex aequali factorum numero constant, si primum factorem $\frac{2}{1}$ producti M per primum factorem $\frac{3}{2}$ producti N, secundum $\frac{4}{3}$ illius, per secundum $\frac{5}{4}$ hujus et ita porro dividamus, fiet

$$\frac{\mathrm{M}}{\mathrm{N}} = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \cdot \text{etc.}$$

unde obtinemus pro casu $x = 1$, per productum infinitum,

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \cdot \text{etc.} = \frac{\pi}{2}.$$

Corollarium 1.

357. Pro valore ergo ipsius π idem productum infinitum elicuimus, quod olim jam Wallisius invenerat, et cujus veritatem

in Introductione confirmavimus, diversissimis viis incedentes, erit itaque

$$\pi = 2 \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \text{etc.}$$

Corollarium 2.

358. Nihil interest, quonam ordine singuli factores in hoc producto disponantur, dummodo nulli relinquantur. Ita aliquot ab initio seorsim sumendo, reliqui ordine debito disponi possunt; veluti

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2 \cdot 4}{1 \cdot 3} \times \frac{2 \cdot 6}{3 \cdot 5} \cdot \frac{4 \cdot 8}{5 \cdot 7} \cdot \frac{6 \cdot 10}{7 \cdot 9} \cdot \frac{8 \cdot 12}{9 \cdot 11} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2}{3} \times \frac{2 \cdot 4}{1 \cdot 5} \cdot \frac{4 \cdot 6}{3 \cdot 7} \cdot \frac{6 \cdot 8}{5 \cdot 9} \cdot \frac{8 \cdot 10}{7 \cdot 11} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2 \cdot 4}{3 \cdot 5} \times \frac{2 \cdot 6}{1 \cdot 7} \cdot \frac{4 \cdot 8}{3 \cdot 9} \cdot \frac{6 \cdot 10}{5 \cdot 11} \cdot \frac{8 \cdot 12}{7 \cdot 13} \cdot \text{etc.}$$

Scholion.

359. Fundamentum ergo hujus evolutionis in hoc consistit, quod valor integralis $\int \frac{x^{i+\alpha}\,\partial x}{\sqrt{(1-xx)}}$, denotante i numerum infinitum, idem sit, utcunque numerus finitus α varietur. Atque hoc quidem ex reductione

$$\int \frac{x^{i-1}\,\partial x}{\sqrt{(1-xx)}} = \frac{i+1}{i} \int \frac{x^{i+1}\,\partial x}{\sqrt{(1-xx)}}$$

manifestum est, si pro α valores binario differentes assumantur. Deinde autem nullum est dubium, quin hoc integrale $\int \frac{x^{i+1}\,\partial x}{\sqrt{(1-xx)}}$ inter haec $\int \frac{x^{i}\,\partial x}{\sqrt{(1-xx)}}$ et $\int \frac{x^{i+2}\,\partial x}{\sqrt{(1-xx)}}$, quasi limites contineatur, qui cum sint inter se aequales necesse est omnes formulas intermedias iisdem quoque esse aequales. Atque hoc latius patet ad formulas magis complicatas, ita ut denotante i numerum infinitum sit

$$\int \frac{x^{i+\alpha}\,\partial x}{(1-x^n)^k} = \int \frac{x^i\,\partial x}{(1-x^n)^k}.$$

Cum enim sit

$$\int \frac{x^{m+n-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}}$$

hae formulae posito $m=\infty$ sunt aequales; unde illarum quoque aequalitas casibus, quibus $\alpha = n$, vel $\alpha = 2n$, vel $\alpha = 3n$ etc. perspicitur; sin autem α medium quempiam valorem teneat formulae, ipsius quoque valor medium quoddam tenere debet inter valores aequales, ideoque ipsis erit aequalis. Hoc igitur principio stabilito sequens problema resolvere poterimus.

Problema 44.

360. Rationem horum duorum integralium

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}} \text{ et } \int x^{\mu-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

casu $x=1$, per productum infinitorum factorum exprimere.

Solutio.

Cum sit

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}} = \frac{m+k}{m}\int x^{m+n-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}};$$

casu $x=1$, valor istius integralis ad integrale infinite remotum reducetur hoc modo:

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}}$$

$$= \frac{(m+k)(m+k+n)(m+k+2n)\ldots\ldots(m+k+in)}{m\quad(m+n)\quad(m+2n)\ldots\ldots(m+in)}\int x^{m+in+n-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

ubi i numerum infinitum denotare assumimus. Simili autem modo pro altera formula proposita erit

$$\int x^{\mu-1}\partial x\,(1-x^n)^{\frac{k-n}{n}}$$

$$=\frac{(\mu+k)(\mu+k+n)(\mu+k+2n)\ldots\ldots(\mu+k+in)}{\mu\quad(\mu+n)\quad(\mu+2n)\ldots\ldots(\mu+in)}\int x^{\mu+in+n-1}\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

atque hae postremae formulae integrales ob exponentes infinitos, aequales erunt, non obstante inaequalitate numerorum m et μ: tum vero bina haec producta infinita pari factorum numero constant. Quare si singuli per singulos, hoc est primus per primum, secundus per secundum dividantur, ratio binorum integralium propositorum ita exprimetur:

$$\frac{\int x^{m-1}\partial x\,(1-x^n)^{\frac{k-n}{n}}}{\int x^{\mu-1}\partial x\,(1-x^n)^{\frac{k-n}{n}}}=\frac{\mu(m+k)}{m(\mu+k)}\cdot\frac{(\mu+n)(m+k+n)}{(m+n)(\mu+k+n)}\cdot\frac{(\mu+2n)(m+k+2n)}{(m+2n)(\mu+k+2n)}\text{ etc.}$$

si quidem ambo integralia ita determinentur, ut posito $x=0$ evanescant, tum vero statuatur $x=1$; litteris autem m, μ, n, k numeros positivos denotari necesse est.

Corollarium 1.

361. Si differentia numerorum m et μ aequetur multiplo ipsius n, in producto invento infiniti factores se destruunt, relinqueturque factorum numerus finitus, uti si $\mu=m+n$ habebitur:

$$\frac{(m+n)(m+k)}{m(m+k+n)}\cdot\frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)}\cdot\frac{(m+3n)(m+k+2n)}{(m+2n)(m+k+3n)}\text{ etc.}$$

quod reducitur ad $\frac{m+k}{m}$.

Corollarium 2.

362. Valor autem illius producti necessario est finitus, id quod tam ex formulis integralibus, quarum rationem exprimit, patet, quam inde, quod in singulis factoribus numeratores et denominatores sunt alternatim majores et minores.

Corollarium 3.

363. Si ponamus $m = 1$, $\mu = 3$, $n = 4$ et $k = 2$, erit

$$\frac{\int \frac{\partial x}{\sqrt{(1 - x^4)}}}{\int \frac{x x \partial x}{\sqrt{(1 - x^4)}}} = \frac{3.3}{1.5} \cdot \frac{7.7}{5.9} \cdot \frac{11.11}{9.13} \cdot \frac{15.15}{13.17} \text{ etc.}$$

supra autem invenimus productum harum binarum formularum esse $= \frac{\pi}{4}$.

Problema 45.

364. Valorem hujus integralis $\int x^{m-1} \partial x (1 - x^n)^{\frac{k-n}{n}}$, quem posito $x = 1$ recipit, per productum infinitum exprimere.

Solutio.

Cum in problemate praecedente ratio hujus integralis ad hoc alterum $\int x^{\mu-1} \partial x (1 - x^n)^{\frac{k-n}{n}}$ per productum infinitum sit assignata, in hoc exponens μ ita accipiatur, ut integrale exhiberi possit. Capiatur ergo $\mu = n$, et integrale fit =

$$C - \frac{1}{k} (1 - x^n)^{\frac{k}{n}} = \frac{1 - (1 - x^n)^{\frac{k}{n}}}{k}$$

ita determinatum, ut posito $x = 0$ evanescat: ponatur nunc, ut conditio postulat, $x = 1$, et quia hoc integrale erit $= \frac{1}{k}$, habebimus fosmulae propositae integrale casu $x = 1$, ita expressum

$$\int x^{m-1} \partial x (1 - x^n)^{\frac{k-n}{n}} = \frac{1}{k} \cdot \frac{n(m+k)}{m(k+n)} \cdot \frac{2n(m+k+n)}{(m+n)(k+2n)} \cdot \frac{3n(m+k+2n)}{(m+2n)(k+3n)} \text{ etc.}$$

quod singulos factores partiendo ita repraesentari potest

$$\int x^{m-1} \partial x (1 - x^n)^{\frac{k-n}{n}} = \frac{n}{mk} \cdot \frac{2n(m+k)}{(m+n)(k+n)} \cdot \frac{3n(m+k+n)}{(m+2n)(k+2n)} \cdot \frac{4n(m+k+2n)}{(m+3n)(k+3n)} \text{ etc.}$$

Corollarium 1.

365. Cum in hac expressione litterae m et k sint permutabiles, sequitur etiam, haec integralia posito $x = 1$ inter se esse aequalia:

$$\int x^{m-1}\partial x\,(1-x^n)^{\frac{k-n}{n}} = \int x^{k-1}\partial x\,(1-x^n)^{\frac{m-n}{n}}$$

quam aequalitatem jam supra §. 349. elicuimus.

Corollarium 2.

366. Cum formulae nostrae valor, si $m = n - k$, aequalis sit valori hujus $\int \frac{z^{k-1}\partial z}{1+z^n}$ posito $z = \infty$, si ob $m + k = n$ statuamus $m = \frac{n+\alpha}{2}$ et $k = \frac{n-\alpha}{2}$, habebimus:

$$\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n+\alpha}{2n}}} \int = \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{n-\alpha}{2n}}} = \int \frac{z^{k-1}\partial z}{1+z^n} = \int \frac{z^{m-1}\partial z}{1+z^n}$$

$$= \frac{4n}{nn-\alpha\alpha}\cdot\frac{2.4nn}{9nn-\alpha\alpha}\cdot\frac{4.6nn}{25nn-\alpha\alpha}\cdot\frac{6.8nn}{49nn-\alpha\alpha}\text{ etc.}$$

Quod productum etiam hoc modo exponi potest

$$\frac{2}{n-\alpha}\cdot\frac{2n.\,2n}{(n+\alpha)(3n-\alpha)}\cdot\frac{4n.\,4n}{(3n+\alpha)(5n-\alpha)}\cdot\frac{6n.\,6n}{(5n+\alpha)(7n-\alpha)}\text{ etc.}$$

quod ergo etiam exprimit valorem ipsius $\frac{\pi}{n\sin.\frac{m\pi}{n}} = \frac{\pi}{n\cos.\frac{\alpha\pi}{2n}}$ per §. 351.

Corollarium 3.

367. Vel si simpliciter ponamus $k = n - m$, fiet

$$\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m}{n}}} = \int \frac{x^{n-m-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}} = \int \frac{z^{m-1}\partial z}{1+z^n} = \int \frac{z^{n-m-1}\partial z}{1+z^n}$$

$$= \frac{1}{n-m} \cdot \frac{nn}{m(2n-m)} \cdot \frac{4nn}{(n+m)(3n-m)} \cdot \frac{9nn}{(2n+m)(4n-m)} \text{ etc.}$$

quae ex forma primum inventa oritur. Haec ergo aequalitas subsistit, si ponatur $x = 1$ et $z = \infty$.

Scholion 1.

368. In Introductione autem pro multiplicatione angulorum inveneram

$$\sin. \frac{m\pi}{n} = \frac{m\pi}{n}\left(1 - \frac{mm}{nn}\right)\left(1 - \frac{mm}{4nn}\right)\left(1 - \frac{mm}{9nn}\right)\left(1 - \frac{mm}{16nn}\right) \text{ etc.}$$

et cum $\sin. \frac{(n-m)\pi}{n} = \sin. \frac{m\pi}{n}$, ob $n - m = k$, erit etiam

$$\sin. \frac{m\pi}{n} = \frac{k\pi}{n}\left(1 - \frac{kk}{nn}\right)\left(1 - \frac{kk}{4nn}\right)\left(1 - \frac{kk}{9nn}\right)\left(1 - \frac{kk}{16nn}\right) \text{ etc.}$$

quae reducitur ad hanc formam

$$\sin. \frac{m\pi}{n} = \frac{k\pi}{n} \cdot \frac{(n-k)(n+k)}{nn} \cdot \frac{(2n-k)(2n+k)}{4nn} \cdot \frac{(3n-k)(3n+k)}{9nn} \text{ etc.}$$

et pro k suo valore restituto

$$\sin. \frac{m\pi}{n} = \frac{\pi}{n}(n-m) \cdot \frac{m(2n-m)}{nn} \cdot \frac{(n+m)(3n-m)}{4nn} \cdot \frac{(2n+m)(4n-m)}{9nn} \text{ etc.}$$

unde manifesto pro $\frac{\pi}{n \sin. \frac{m\pi}{n}}$ idem reperitur productum, quod valorem nostrorum integralium exprimit, sicque novam habemus demonstrationem pro Theoremate illo eximio supra per multas ambages evicto, esse

$$\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m}{n}}} = \int \frac{x^{n-m-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}} = \int \frac{z^{m-1}\partial z}{1+z^n} = \int \frac{z^{n-m-1}\partial z}{1+z^n}$$

$$= \frac{\pi}{n \sin. \frac{m\pi}{n}}.$$

Scholion 2.

369. Quo nostra formula latius pateat, ponamus $\frac{k}{n} = \frac{\mu}{\nu}$ seu $k = \frac{\mu n}{\nu}$, et nanciscemur $\int x^{m-1}\partial x\,(1-x^n)^{\frac{\mu}{\nu}-1}$

$$= \frac{\nu}{m\mu} \cdot \frac{2(m\nu + n\mu)}{(m+n)(\mu+\nu)} \cdot \frac{3[m\nu + n(\mu+\nu)]}{(m+2n)(\mu+2\nu)} \cdot \frac{4[m\nu + n(\mu+2\nu)]}{(m+3n)(\mu+3\nu)} \cdot \text{etc.}$$
$$= \frac{\nu}{m\mu} \cdot \frac{2(m\nu + n\mu)}{(m+n)(\mu+\nu)} \cdot \frac{3(m\nu + n\mu + n\nu)}{(m+2n)(\mu+2\nu)} \cdot \frac{4(m\nu + n\mu + 2n\nu)}{(m+3n)(\mu+3\nu)} \cdot \frac{5(m\nu + n\mu + 3n\nu)}{(m+4n)(\mu+4\nu)} \text{ etc.}$$

in qua expressione litterae m, n et μ, ν sunt permutabiles, praeterquam in primo factore, qui cum reliquis lege continuitatis non connectitur; ac si per n multiplicemus, permutabilitas erit perfecta, unde concludimus fore

$$n \int x^{m-1} \partial x (1 - x^n)^{\frac{\mu}{\nu} - 1} = \nu \int x^{\mu-1} \partial x (1 - x^\nu)^{\frac{m}{n} - 1}$$

quae aequalitas casu $\nu = n$ ad supra observatam reducitur. Caeterum juvabit casus praecipuos perpendisse, quos ex valoribus μ et ν desumamus.

Exemplum 1.

370. *Sit* $\mu = 1$ *et* $\nu = 2$, *fietque*

$$\int \frac{x^{m-1} \partial x}{\sqrt{(1 - x^n)}} = \frac{2}{m} \cdot \frac{2(2m+n)}{3(m+n)} \cdot \frac{3(2m+3n)}{5(m+2n)} \cdot \frac{4(2m+5n)}{7(m+3n)} \text{ etc.}$$
$$= \frac{2}{n} \int \frac{\partial x}{\sqrt[n]{(1 - x^2)^{n-m}}}$$

quae expressio ita commodius repraesentatur:

$$\int \frac{x^{m-1} \partial x}{\sqrt{(1 - x^n)}} = \frac{2}{m} \cdot \frac{4(2m+n)}{3(2m+2n)} \cdot \frac{6(2m+3n)}{5(2m+4n)} \cdot \frac{8(2m+5n)}{7(2m+6n)} \text{ etc.}$$

unde sequentes casus specialissimi deducuntur:

$$\int \frac{\partial x}{\sqrt{(1 - xx)}} = 2 \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \text{ etc.} \quad = \int \frac{\partial x}{\sqrt{(1 - xx)}}$$
$$\int \frac{\partial x}{\sqrt{(1 - x^3)}} = 2 \cdot \frac{4 \cdot 5}{3 \cdot 8} \cdot \frac{6 \cdot 11}{5 \cdot 14} \cdot \frac{8 \cdot 17}{7 \cdot 20} \cdot \frac{10 \cdot 23}{9 \cdot 26} \text{ etc.} = \frac{2}{3} \int \frac{\partial x}{\sqrt[3]{(1 - x^2)^2}}$$
$$\int \frac{x \partial x}{\sqrt{(1 - x^3)}} = 1 \cdot \frac{4 \cdot 7}{3 \cdot 10} \cdot \frac{6 \cdot 13}{5 \cdot 16} \cdot \frac{8 \cdot 19}{7 \cdot 22} \cdot \frac{10 \cdot 25}{9 \cdot 28} \text{ etc.} = \frac{2}{3} \int \frac{\partial x}{\sqrt[3]{(1 - x^2)}}$$

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = 2 \cdot \frac{4.5}{3.5} \cdot \frac{6.7}{5.9} \cdot \frac{8.11}{7.13} \cdot \frac{10.15}{9.17} \text{ etc.} = \tfrac{1}{2} \int \frac{\partial x}{\sqrt[4]{(1-x^2)^3}}$$

$$\int \frac{x \partial x}{\sqrt{(1-x^4)}} = 1 \cdot \frac{4.4}{3.6} \cdot \frac{6.8}{5.10} \cdot \frac{8.12}{7.14} \cdot \frac{10.16}{9.18} \text{ etc.} = \tfrac{1}{2} \int \frac{\partial x}{\sqrt{(1-x^2)}}$$

$$\text{sive} \quad = 1 \cdot \frac{2.4}{3.3} \cdot \frac{4.6}{5.5} \cdot \frac{6.8}{7.7} \cdot \frac{8.10}{9.9} \text{ etc.}$$

$$\int \frac{x x \partial x}{\sqrt{(1-x^4)}} = \tfrac{2}{3} \cdot \frac{4.5}{3.7} \cdot \frac{6.9}{5.11} \cdot \frac{8.13}{7.15} \cdot \frac{10.17}{9.19} \text{ etc.} = \tfrac{1}{2} \int \frac{\partial x}{\sqrt[4]{(1-x^2)}}$$

$$\int \frac{x^3 \partial x}{\sqrt{(1-x^4)}} = \tfrac{2}{4} \cdot \frac{4.6}{3.8} \cdot \frac{6.10}{5.12} \cdot \frac{8.14}{7.16} \cdot \frac{10.18}{9.20} \text{ etc.} = \tfrac{1}{2}$$

Exemplum 2.

371. *Sit* $\mu = 1$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1} \partial x}{\sqrt[3]{(1-x^n)^2}} = \frac{3}{m} \cdot \frac{2(3m+n)}{4(m+n)} \cdot \frac{3(3m+4n)}{7(m+2n)} \cdot \frac{4(3m+7n)}{10(m+3n)} \text{ etc.}$$

$$= \frac{3}{n} \int \frac{\partial x}{\sqrt[n]{(1-x^3)^{n-m}}}:$$

unde sequentes casus specialissimi deducuntur:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^2)^2}} = \tfrac{3}{1} \cdot \frac{2.5}{4.3} \cdot \frac{3.11}{7.5} \cdot \frac{4.17}{10.7} \cdot \frac{5.23}{13.9} \text{ etc.} = \tfrac{3}{2} \int \frac{\partial x}{\sqrt{(1-x^3)}}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3}{1} \cdot \frac{2.6}{4.4} \cdot \frac{3.15}{7.7} \cdot \frac{4.24}{10.10} \cdot \frac{5.33}{13.13} \text{ etc.} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{sive} \quad = \tfrac{3}{1} \cdot \frac{2.6}{4.4} \cdot \frac{5.9}{7.7} \cdot \frac{8.12}{10.10} \cdot \frac{11.15}{13.13} \text{ etc.}$$

$$\int \frac{x \partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3}{2} \cdot \frac{2.9}{4.5} \cdot \frac{3.18}{7.8} \cdot \frac{4.27}{10.11} \cdot \frac{5.36}{13.14} \text{ etc.} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\text{sive} \quad = \tfrac{3}{2} \cdot \frac{3.6}{4.5} \cdot \frac{6.9}{7.8} \cdot \frac{9.12}{10.11} \cdot \frac{12.15}{13.14} \text{ etc.}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^4)^2}} = \tfrac{3}{1} \cdot \frac{2.7}{4.5} \cdot \frac{3.19}{7.9} \cdot \frac{4.31}{10.13} \cdot \frac{5.43}{13.17} \text{ etc.} = \tfrac{3}{4} \int \frac{\partial x}{\sqrt[4]{(1-x^3)^3}}$$

$$\int \frac{x x \partial x}{\sqrt[3]{(1-x^4)^2}} = 1 \cdot \frac{2.13}{4.7} \cdot \frac{3.25}{7.11} \cdot \frac{4.37}{10.15} \cdot \frac{5.49}{13.19} \text{ etc.} = \tfrac{3}{4} \int \frac{\partial x}{\sqrt[4]{(1-x^3)}}$$

Exemplum 3.

372. *Sit* $\mu = 2$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1}\,\partial x}{\sqrt[3]{(1-x^n)}} = \frac{3}{2m} \cdot \frac{2(3m+2n)}{5(m+n)} \cdot \frac{3(3m+5n)}{8(m+2n)} \cdot \frac{4(3m+8n)}{11(m+3n)} \text{ etc.}$$

$$= \frac{3}{n} \int \frac{x\,\partial x}{\sqrt[n]{(1-x^3)^{n-m}}}:$$

unde sequentes casus speciales deducuntur:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^2)}} = \frac{3}{2} \cdot \frac{2.\ 7}{5.\ 3} \cdot \frac{3.\ 13}{8.\ 5} \cdot \frac{4.\ 19}{11.\ 7} \cdot \frac{5.\ 25}{14.\ 9} \text{ . etc.} = \frac{3}{2} \int \frac{x\,\partial x}{\sqrt{(1-x^3)}}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \frac{3}{2} \cdot \frac{2.\ 9}{5.\ 4} \cdot \frac{3.\ 18}{8.\ 7} \cdot \frac{4.\ 27}{11.\ 10} \cdot \frac{5.\ 36}{14.\ 13} \text{ . etc.} = \int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{sive} \quad = \frac{3.\ 3}{2.\ 4} \cdot \frac{6.\ 6}{5.\ 7} \cdot \frac{9.\ 9}{8.\ 10} \cdot \frac{12.\ 12}{11.\ 13} \text{ . etc.}$$

$$\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}} = \frac{3}{4} \cdot \frac{2.\ 12}{5.\ 5} \cdot \frac{3.\ 21}{8.\ 8} \cdot \frac{4.\ 30}{11.\ 11} \cdot \frac{5.\ 39}{14.\ 14} \text{ . etc.} = \int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\text{sive} \quad = \frac{3}{4} \cdot \frac{4.\ 6}{5.\ 5} \cdot \frac{7.\ 9}{8.\ 8} \cdot \frac{10.\ 12}{11.\ 11} \cdot \frac{13.\ 15}{14.\ 14} \text{ . etc.}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^4)}} = \frac{3}{2} \cdot \frac{2.\ 11}{5.\ 5} \cdot \frac{3.\ 23}{8.\ 9} \cdot \frac{4.\ 35}{11.\ 13} \cdot \frac{5.\ 47}{14.\ 17} \text{ . etc.} = \frac{3}{4} \int \frac{x\,\partial x}{\sqrt[4]{(1-x^3)^3}}$$

$$\int \frac{x x\,\partial x}{\sqrt[3]{(1-x^4)}} = \frac{1}{2} \cdot \frac{2.\ 17}{5.\ 7} \cdot \frac{3.\ 29}{8.\ 11} \cdot \frac{4.\ 41}{11.\ 15} \cdot \frac{5.\ 53}{14.\ 19} \text{ . etc.} = \frac{3}{4} \int \frac{x\,\partial x}{\sqrt[4]{(1-x^3)}}.$$

Exemplum 4.

373. *Sit* $\mu = 1$ *et* $\nu = 4$, *fietque*

$$\int \frac{x^{m-1}\,\partial x}{\sqrt[4]{(1-x^n)^3}} = \frac{4}{m} \cdot \frac{2(4m+n)}{5(m+n)} \cdot \frac{3(4m+5n)}{9(m+2n)} \cdot \frac{4(4m+9n)}{13(m+3n)} \text{ etc.}$$

$$= \frac{4}{n} \int \frac{\partial x}{\sqrt[n]{(1-x^4)^{n-m}}}:$$

unde sequentes casus speciales prodeunt:

$$\int\frac{\partial x}{\sqrt[4]{(1-x^2)^3}} = \frac{4}{1}\cdot\frac{2.6}{5.3}\cdot\frac{3.14}{9.5}\cdot\frac{4.22}{13.7}\cdot\frac{5.30}{17.9}\,.\,\text{etc.} = 2\int\frac{\partial x}{\sqrt{(1-x^4)}}$$

$$\text{seu} \qquad = \frac{4}{1}\cdot\frac{4.3}{3.5}\cdot\frac{6.7}{5.9}\cdot\frac{8.11}{7.13}\cdot\frac{10.15}{9.17}\,.\,\text{etc.}$$

$$\int\frac{\partial x}{\sqrt[4]{(1-x^3)^3}} = \frac{4}{1}\cdot\frac{2.7}{5.4}\cdot\frac{3.19}{9.7}\cdot\frac{4.31}{13.10}\cdot\frac{5.43}{17.13}\,.\,\text{etc.} = \frac{4}{3}\int\frac{\partial x}{\sqrt[3]{(1-x^4)^2}}$$

$$\int\frac{x\,\partial x}{\sqrt[4]{(1-x^3)^3}} = \frac{2}{1}\cdot\frac{2.11}{5.5}\cdot\frac{3.23}{9.8}\cdot\frac{4.35}{13.11}\cdot\frac{5.47}{17.14}\,.\,\text{etc.} = \frac{4}{3}\int\frac{\partial x}{\sqrt[3]{(1-x^4)}}$$

$$\int\frac{\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{1}\cdot\frac{2.8}{5.5}\cdot\frac{3.24}{9.9}\cdot\frac{4.40}{13.13}\cdot\frac{5.56}{17.17}\,.\,\text{etc.} = \int\frac{\partial x}{\sqrt[4]{(1-x^4)^3}}$$

$$\text{seu} \qquad = \frac{4}{1}\cdot\frac{4.4}{5.5}\cdot\frac{6.12}{9.9}\cdot\frac{8.20}{13.13}\cdot\frac{10.28}{17.17}\,.\,\text{etc.}$$

$$\text{seu} \qquad = \frac{4}{1}\cdot\frac{2.8}{5.5}\cdot\frac{6.12}{9.9}\cdot\frac{10.16}{13.13}\cdot\frac{14.20}{17.17}\,.\,\text{etc.}$$

$$\int\frac{x\,x\,\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{3}\cdot\frac{2.16}{5.7}\cdot\frac{3.32}{9.11}\cdot\frac{4.48}{13.15}\cdot\frac{5.64}{17.19}\,.\,\text{etc.} = \int\frac{\partial x}{\sqrt[4]{(1-x^4)}}$$

$$\text{seu} \qquad = \frac{4}{3}\cdot\frac{4.8}{5.7}\cdot\frac{6.16}{9.11}\cdot\frac{8.24}{13.15}\cdot\frac{10.32}{17.19}\,.\,\text{etc.}$$

$$\text{seu} \qquad = \frac{4}{3}\cdot\frac{4.8}{5.7}\cdot\frac{8.12}{9.11}\cdot\frac{12.16}{13.15}\cdot\frac{16.20}{17.17}\,.\,\text{etc.}$$

Atque in his et praecedentibus jam casus $\mu = 3$ et $\nu = 4$ est contentus.

Scholion.

374. Caeterum hae formulae, in quas litteras μ et ν introduxi, latius non patent quam primum consideratae, series enim pendent a binis fractionibus $\frac{m}{n}$ et $\frac{\mu}{\nu}$, quae cum semper ad communem denominatorem revocari queant, formulas

$$\int\frac{x^{m-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-k}}} = \int\frac{x^{k-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}$$

perpendisse sufficiet. Cum igitur earum valor casu $x = 1$ aequetur huic producto

$$\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}\,.\,\text{etc.}$$

si in singulis membris factores numeratorum permutemus, et membra aliter partiamur, idem productum hanc induet formam

$$\frac{m+k}{mk} \cdot \frac{n(m+k+n)}{(m+n)(k+n)} \cdot \frac{2n(m+k+2n)}{(m+2n)(k+2n)} \cdot \frac{3n(m+k+3n)}{(m+3n)(k+3n)} \cdot \text{etc.}$$

quae ad memoriam magis accommodata videtur. Simili modo cum sit:

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

$$= \frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \cdot \frac{2n(p+q+2n)}{(p+2n)(q+2n)} \cdot \frac{3n(p+q+3n)}{(p+3n)(q+3n)} \cdot \text{etc.}$$

illam formam per hanc dividendo, erit

$$\frac{\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}}}{\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}}}$$

$$= \frac{pq(m+k)}{mk(p+q)} \cdot \frac{(p+n)(q+n)(m+k+n)}{(m+n)(k+n)(p+q+n)} \cdot \frac{(p+2n)(q+2n)(m+k+2n)}{(m+2n)(k+2n)(p+q+2n)} \cdot \text{etc.}$$

cujus omnia membra eadem lege continentur. Hinc autem eximiae comparationes hujusmodi formularum deduci possunt, quae quo facilius commemorari queant, brevitatis causa sequenti scriptionis compendio utar.

Definitio.

375. Formulae integralis $\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}}$ valorem, quem posito $x=1$ recipit, brevitatis gratia hoc signo $\left(\frac{p}{q}\right)$ indicemus, ubi quidem exponentem n, quem in comparatione plurium hujusmodi formularum eundem esse assumo, subintelligi oportet.

Corollarium 1.

376. Primum igitur patet esse $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$, et utramque formulam esse

$$= \frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \cdot \frac{2n(p+q+2n)}{(p+2n)(q+2n)} \cdot \text{etc.}$$

quorum membrorum progressio est manifesta; dum singuli factores tam numeratoris quam denominatoris continuo eodem numero n augentur, ita ut ex cognito primo membro sequentia facile formentur.

Corollarium 2.

377. Deinde si sit $p = n$, ob formulam integrabilem liquet esse $\left(\frac{n}{q}\right) = \left(\frac{q}{n}\right) = \frac{1}{q}$, item $\left(\frac{p}{n}\right) = \left(\frac{n}{p}\right) = \frac{1}{p}$. Porro cum

$$\int x^{p-1} \partial x \, (1 - x^n)^{-\frac{p}{n}} = \frac{\pi}{n \sin. \frac{p\pi}{n}},$$

ob $q - n = -p$ seu $p + q = n$ erit

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \frac{\pi}{n \sin. \frac{p\pi}{n}}.$$

Quare valor formulae $\left(\frac{p}{q}\right)$ absolute assignari potest, quoties fuerit vel $p = n$, vel $q = n$, vel $p + q = n$.

Corollarium 3.

378. Quia etiam invenimus hanc reductionem

$$\int x^{p+n-1} \partial x \, (1 - x^n)^{\frac{q-n}{n}} = \frac{p}{p+q} \int x^{p-1} \partial x \, (1 - x^n)^{\frac{q-n}{n}},$$

sequitur fore $\left(\frac{p+n}{q}\right) = \frac{p}{p+q} \cdot \left(\frac{p}{q}\right)$: hincque

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = \frac{p-n}{p+q-n} \left(\frac{p-n}{q}\right) = \frac{q-n}{p+q-n} \left(\frac{p}{q-n}\right);$$

tum vero etiam

$$\left(\frac{p}{q}\right) = \frac{(p-n)(q-n)}{(p+q-n)(p+q-2n)} \cdot \left[\frac{p-n}{q-n}\right]:$$

unde semper numeri p et q infra n deprimi possunt.

Problema 46.

379. Invenire diversa producta ex binis hujusmodi formulis, quae inter se sint aequalia.

Solutio.

Quaerantur ergo numeri a, b, c, d, et p, q, r, s, ut fiat $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{p}{q}\right)\left(\frac{r}{s}\right)$, quod cum sit

$$\left(\frac{a}{b}\right) = \frac{a+b}{ab} \cdot \frac{n(a+b+n)}{(a+n)(b+n)} \text{ etc. } \left(\frac{c}{d}\right) = \frac{c+d}{cd} \cdot \frac{n(c+d+n)}{(c+n)(d+n)} \text{ etc.}$$
$$\left(\frac{p}{q}\right) = \frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \text{ etc. } \left(\frac{r}{s}\right) = \frac{r+s}{rs} \cdot \frac{n(r+s+n)}{(r+n)(s+n)} \text{ etc.}$$

eveniet, si fuerit

$$\frac{(a+b)(c+d)}{abcd} = \frac{(p+q)(r+s)}{pqrs}, \text{ seu}$$
$$abcd\,(p+q)(r+s) = pqrs\,(a+b)(c+d)$$

ita ut, cum utrinque sex sint factores, singuli singulis sint aequales. Ex quaternis ergo $abcd$ et $pqrs$ binos ad minimum aequales esse oportet; sit itaque $s = d$ efficique oportet

$$abc\,(p+q)(r+d) = pqr\,(a+b)(c+d).$$

I. Sumatur alter factor r, qui cum ipsi c aequari nequeat, quia alioquin fieret $\left(\frac{c}{d}\right) = \left(\frac{r}{s}\right)$, statuatur $r = b$, ut fiat

$$ac\,(p+q)(b+d) = pq\,(a+b)(c+d).$$

Hic neque p neque q ipsi $p+q$ aequari potest, poni ergo debet:

1.) Vel $p+q = a+b$, ut sit $ac\,(b+d) = pq\,(c+d)$ quia neque c neque $(b+d)$ ipsi $(c+d)$ aequari potest, fieret enim vel $d = 0$, vel $b = c$, et $\left(\frac{r}{s}\right) = \left(\frac{c}{d}\right)$, relinquitur $a = c+d$, et $pq = c\,(b+d)$; ideoque $p = b+d$ et $q = c$, unde conficitur:

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{b+d}{c}\right)\left(\frac{b}{d}\right).$$

2.) Vel $p+q = c+d$, ergo $ac\,(b+d) = pq\,(a+b)$, hic c neque ipsi p neque q aequari potest, fieret enim $\left(\frac{p}{q}\right) = \left(\frac{c}{d}\right)$, unde fiat $c = a+b$, ut sit $pq = a\,(b+d)$; ergo $p = a$; $q = b+d$; $r = b$; $s = d$ consequenter

$$\left(\frac{a}{b}\right)\left(\frac{a+b}{d}\right)=\left(\frac{b+d}{a}\right)\left(\frac{b}{d}\right).$$

II. Quia $r=b$ non differt a praecedenti ob a et b permutabiles, statuatur $r=p+q$, fietque

$$abc\,(d+p+q)=pq\,(a+b)\,(c+d).$$

Quoniam r ipsi c aequari nequit, factor $d+p+q$ neque ipsi p, neque q, neque $c+d$ aequalis poni potest, relinquitur ergo $d+p+q=a+b$, et $abc=pq\,(c+d)$, ubi quia c ipsi $c+d$ aequari nequit, ac p et q pari conditione gaudent, fiat $p=c$; erit $q=a+b-c-d$, et $ab=(c+d)\,(a+b-c-d)$; unde $a=c+d$; $q=b$; $p=c$; $r=b+c$; $s=d$; sicque conficitur:

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right).$$

Corollarium 1.

380. Hae solutiones eodem fere redeunt, indeque tria producta binarum formularum, aequalia eruuntur:

$$\left(\frac{c}{d}\right)\left(\frac{c+d}{b}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right)=\left(\frac{b}{d}\right)\left(\frac{b+d}{c}\right)$$

vel in litteris p, q, r,

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right).$$

Corollarium 2.

381. Si hae formulae in producta infinita evolvantur, reperietur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn\,(p+q+r+n)}{(p+n)\,(q+n)\,(r+n)}\cdot\frac{4nn\,(p+q+r+2n)}{(p+2n)\,(q+2n)\,(r+2n)}\ \text{etc.}$$

unde patet, tres litteras p, q, r, utcunque inter se permutari posse, atque hinc ternas illas formulas concludere licet.

Corollarium 3.

382. Restituamus ipsas formulas integrales, et sequentia tria producta erunt inter se aequalia

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \cdot \int \frac{x^{p+q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} =$$

$$\int \frac{x^{q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{q+r-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}} =$$

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Corollarium 4.

383. Hic casus notatu dignus, quo $p+q=n$, tum enim ob

$$\left(\frac{p+q}{r}\right)=\left(\frac{n}{r}\right)=\frac{1}{r} \text{ et } \left(\frac{p}{q}\right)=\frac{\pi}{n \sin. \frac{p\pi}{n}},$$

haec tria producta fient $=\dfrac{\pi}{nr \sin. \frac{p\pi}{n}}$. Erit scilicet

$$\int \frac{x^{n-p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{n-p+r-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}} = \int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1}\,\partial x}{\sqrt[n]{(1-x^n)^{p}}}$$

$$= \frac{\pi}{nr \sin. \frac{p\pi}{n}}.$$

Scholion.

384. Triplex ista proprietas productorum ex binis formulis maxime est notatu digna, ac pro variis numeris loco p, q, r substituendis obtinebuntur sequentes aequalitates speciales:

p	q	r	
1	1	2	$(\frac{1}{1})(\frac{2}{2}) = (\frac{2}{1})(\frac{3}{1})$
1	2	2	$(\frac{2}{1})(\frac{3}{2}) = (\frac{2}{2})(\frac{4}{1})$
1	2	3	$(\frac{2}{1})(\frac{3}{3}) = (\frac{3}{2})(\frac{5}{1}) = (\frac{3}{1})(\frac{4}{2})$
1	1	3	$(\frac{1}{1})(\frac{3}{2}) = (\frac{3}{1})(\frac{4}{1})$
2	2	3	$(\frac{2}{2})(\frac{4}{3}) = (\frac{3}{2})(\frac{5}{2})$
1	3	3	$(\frac{3}{1})(\frac{4}{3}) = (\frac{3}{3})(\frac{6}{1})$
2	3	3	$(\frac{3}{2})(\frac{5}{3}) = (\frac{3}{3})(\frac{6}{2})$
1	1	4	$(\frac{1}{1})(\frac{4}{2}) = (\frac{4}{1})(\frac{5}{1})$
1	2	4	$(\frac{2}{1})(\frac{4}{3}) = (\frac{4}{2})(\frac{6}{1}) = (\frac{4}{1})(\frac{5}{2})$
1	3	4	$(\frac{3}{1})(\frac{4}{4}) = (\frac{4}{1})(\frac{5}{3}) = (\frac{4}{3})(\frac{7}{1})$
1	4	4	$(\frac{4}{1})(\frac{5}{4}) = (\frac{4}{4})(\frac{8}{1})$
2	2	4	$(\frac{2}{2})(\frac{4}{4}) = (\frac{4}{2})(\frac{6}{2})$
2	3	4	$(\frac{3}{2})(\frac{5}{4}) = (\frac{4}{3})(\frac{7}{2}) = (\frac{4}{2})(\frac{6}{3})$
2	4	4	$(\frac{4}{2})(\frac{6}{4}) = (\frac{4}{4})(\frac{8}{2})$
3	3	4	$(\frac{3}{3})(\frac{6}{4}) = (\frac{4}{3})(\frac{7}{3})$
3	4	4	$(\frac{4}{3})(\frac{7}{4}) = (\frac{4}{4})(\frac{8}{3})$.

Quae formulae pro omnibus numeris n valent, ac si numeri majores quam n occurrant, eos ad minores reduci posse supra vidimus.

Problema 47.

385. Invenire producta diversa ex ternis hujusmodi formulis, quae inter se sint aequalia.

Solutio.

Consideretur productum $(\frac{p}{q})(\frac{p+q}{r})(\frac{p+q+r}{s})$, quod evolutum praebet:

$$\frac{p+q+r+s}{pqrs} \cdot \frac{n3\,(p+q+r+s+n)}{(p+n)\,(q+n)\,(r+n)\,(s+n)} \text{ etc.}$$

quod eundem valorem retinere evidens est, quomodocunque quatuor litterae inter se commutentur. Tum vero eadem evolutio prodit ex hoe producto: $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)$, ubi eadem permutatio locum habet. Aequalia ergo sunt inter se omnia haec producta:

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{q}\right)\left(\frac{p+q+s}{r}\right);$$
$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{p}{q}\right)\left(\frac{p+r}{s}\right)\left(\frac{p+r+s}{q}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{r}\right)\left(\frac{p+r+s}{q}\right);$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{p}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right);$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{s}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{r}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{q}\right)\left(\frac{q+r+s}{p}\right).$$

Producta alterius formae ope praecedentis proprietatis hinc sponte fluunt: est enim

$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Deinde vero etiam hoc productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)$ evolutum pro primo membro dat: $\frac{(p+q+r)(p+r+s)}{pqrs(p+r)}$, in quo tam p et r, quam q et s inter se permutare licet, ita ut sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

Scholion.

386. Quantumvis late haec patere videantur, tamen nullas novas comparationes suppeditant, quae non jam in praecedenti contineantur. Postrema enim aequalitas

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right)$$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)$$
$$\left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right).$$

Priorum vero formatio ex hoc exemplo patebit, aequalitas $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$$
$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Istae autem comparationes praecipue utiles sunt ad valores diversarum formularum ejusdem ordinis seu pro dato numero n invicem reducendos, ut integratio ad paucissimas revocetur, quibus datis reliquae per eas definiri queant.

Problema 48.

387. Formulas simplicissimas exhibere, ad quas integratio omnium casuum in forma $\left(\frac{p}{q}\right) = \int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$ contentorum reduci queat.

Solutio.

Primo est $\left(\frac{n}{p}\right) = \frac{1}{p}$, unde habentur hi casus

$$\left(\frac{n}{1}\right) = 1;\ \left(\frac{n}{2}\right) = \tfrac{1}{2};\ \left(\frac{n}{3}\right) = \tfrac{1}{3};\ \left(\frac{n}{4}\right) = \tfrac{1}{4};\ \left(\frac{n}{5}\right) = \tfrac{1}{5} \text{ etc.}$$

Deinde est $\left(\frac{p}{n-p}\right) = \frac{\pi}{n \sin. \frac{p\pi}{n}}$, unde omnium harum formularum valores sunt cogniti, quas indicemus:

$$\left(\frac{n-1}{1}\right) = \alpha;\ \left(\frac{n-2}{2}\right) = \beta;\ \left(\frac{n-3}{3}\right) = \gamma;\ \left(\frac{n-4}{4}\right) = \delta \text{ etc.}$$

Verum hi non sufficiunt ad reliquos omnes expediendos, praeterea tanquam cognitos spectari oportet hos:

$$\left(\frac{n-2}{1}\right) = A;\ \left(\frac{n-3}{2}\right) = B;\ \left(\frac{n-4}{3}\right) = C;\ \left(\frac{n-5}{4}\right) = D \text{ etc.}$$

atque ex his reliqui omnes determinari poterunt ope aequationum supra demonstratarum; unde potissimum has notasse juvabit:

$$\left(\frac{n-a}{a}\right)\left(\frac{n}{b}\right) = \left(\frac{n-a}{b}\right)\left(\frac{n-a+b}{a}\right);$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-a-b}{b}\right) = \left(\frac{n-b}{b}\right)\left(\frac{n-a-b}{a}\right);$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-b-1}{b}\right)\left(\frac{n-a-b}{a-1}\right) = \left(\frac{n-b}{b}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-b}{a}\right).$$

Ex harum prima posito $a = b + 1$ invenitur

$$\left(\frac{n-1}{a}\right)=\left(\frac{n-a}{a}\right)\left(\frac{n}{a-1}\right):\left(\frac{n-a}{a-1}\right),$$

ubi $\left(\frac{n}{a-1}\right)=\frac{1}{a-1}$, ideoque per formulas assumtas definitur $\left(\frac{n-1}{a}\right)$.

Ex secunda posito $b=1$ deducitur

$$\left(\frac{n-a-1}{1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right).$$

Ex tertia posito $b=1$ invenitur

$$\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

sicque reperiuntur omnes formulae $\left(\frac{n-a-2}{a}\right)$, et ex his porro ponendo $b=2$ in tertia

$$\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)$$

unde reperiuntur formae $\left(\frac{n-a-3}{a}\right)$, et ita porro omnes $\left(\frac{n-a-b}{a}\right)$, quippe quae forma omnes complectitur. Labor autem per priores aequationes non mediocriter contrahitur. Inventa enim $\left(\frac{n-a-2}{a}\right)$ ex prima colligitur

$$\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right)$$

ex secunda vero

$$\left(\frac{n-a-2}{3}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)$$

similique modo ex inventis formulis $\left(\frac{n-a-3}{a}\right)$ derivantur hae

$$\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right)$$

$$\left(\frac{n-a-3}{3}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-a}{a}\right).$$

Corollarium 1.

388. Ex aequatione $\left(\frac{n-1}{a}\right)=\frac{1}{a-1}\left(\frac{n-a}{a}\right):\left(\frac{n-a}{a-1}\right)$ definiuntur

$$\left(\frac{n-1}{2}\right)=\frac{\beta}{1\,A};\ \left(\frac{n-1}{3}\right)=\frac{\gamma}{2\,B};\ \left(\frac{n-1}{4}\right)=\frac{\delta}{3\,C};\ \left(\frac{n-1}{5}\right)=\frac{\varepsilon}{4\,D};\ \text{etc.}$$

Ex aequatione vero $\left(\frac{n-a-1}{1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)$ hae formulae

praeterea vero una transcendente singulari opus est $\left(\frac{2}{1}\right) = A$, und
reliquae ita determinantur:

$$\left(\tfrac{4}{1}\right) = 1;\ \left(\tfrac{4}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{4}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{4}{4}\right) = \tfrac{1}{4}$$
$$\left(\tfrac{3}{1}\right) = \alpha;\ \left(\tfrac{3}{2}\right) = \tfrac{\beta}{A};\ \left(\tfrac{3}{3}\right) = \tfrac{\alpha}{2A}$$
$$\left(\tfrac{2}{1}\right) = A;\ \left(\tfrac{2}{2}\right) = \beta$$
$$\left(\tfrac{1}{1}\right) = \tfrac{\alpha A}{\beta}.$$

Exemplum 4.

394. *Casus in hac forma* $\int \frac{x^{p-1}\,\partial x}{\sqrt[5]{(1-x^5)^{5-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* n $= 5$, *evolvere, ubi est* $\left(\frac{p+5}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae formulae:

$$\left(\tfrac{4}{1}\right) = \frac{\pi}{5 \sin. \frac{\pi}{5}} = \alpha \text{ et } \left(\tfrac{3}{2}\right) = \frac{\pi}{5 \sin. \frac{2\pi}{5}} = \beta,$$

praeter quas duas novas transcendentes assumi oportet

$$\left(\tfrac{3}{1}\right) = A \text{ et } \left(\tfrac{2}{2}\right) = B,$$

per quas omnes sequenti modo determinantur

$$\left(\tfrac{5}{1}\right) = 1;\ \left(\tfrac{5}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{5}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{5}{4}\right) = \tfrac{1}{4};\ \left(\tfrac{5}{5}\right) = \tfrac{1}{5}$$
$$\left(\tfrac{4}{1}\right) = \alpha;\ \left(\tfrac{4}{2}\right) = \tfrac{\beta}{A};\ \left(\tfrac{4}{3}\right) = \tfrac{\beta}{2B};\ \left(\tfrac{4}{4}\right) = \tfrac{\alpha}{3A};$$
$$\left(\tfrac{3}{1}\right) = A;\ \left(\tfrac{3}{2}\right) = \beta;\ \left(\tfrac{3}{3}\right) = \tfrac{\beta\beta}{\alpha B}$$
$$\left(\tfrac{2}{1}\right) = \tfrac{\alpha B}{\beta};\ \left(\tfrac{2}{2}\right) = B$$
$$\left(\tfrac{1}{1}\right) = \tfrac{\alpha A}{\beta}$$

Exemplum 5.

395. *Casus in hac forma* $\int \frac{x^{p-1}\,\partial x}{\sqrt[6]{(1-x^6)^{6-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* n $= 6$, *evolvere.*

A circulo pendent hae tres formulae:

$$\left(\tfrac{5}{1}\right) = \frac{\pi}{6 \sin. \frac{\pi}{6}} = \frac{\pi}{3} = \alpha;\ \left(\tfrac{4}{2}\right) = \frac{\pi}{6 \sin. \frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = \beta;$$

$$\left(\tfrac{3}{3}\right) = \frac{\pi}{6 \sin. \frac{3\pi}{6}} = \frac{\pi}{6} = \gamma$$

tum vero assumantur hae duae transcendentes;

$$\left(\tfrac{4}{1}\right) = A \text{ et } \left(\tfrac{3}{2}\right) = B$$

atque per has omnes sequenti modo determinantur

$$\left(\tfrac{6}{1}\right) = 1;\ \left(\tfrac{6}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{6}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{6}{4}\right) = \tfrac{1}{4};\ \left(\tfrac{6}{5}\right) = \tfrac{1}{5};\ \left(\tfrac{6}{6}\right) = \tfrac{1}{6}$$

$$\left(\tfrac{5}{1}\right) = \alpha;\ \left(\tfrac{5}{2}\right) = \frac{\beta}{A};\ \left(\tfrac{5}{3}\right) = \frac{\gamma}{2B};\ \left(\tfrac{5}{4}\right) = \frac{\beta}{3B};\ \left(\tfrac{5}{5}\right) = \frac{\alpha}{4A}$$

$$\left(\tfrac{4}{1}\right) = A;\ \left(\tfrac{4}{2}\right) = \beta;\ \left(\tfrac{4}{3}\right) = \frac{\beta\gamma}{\alpha B};\ \left(\tfrac{4}{4}\right) = \frac{\beta\gamma A}{2\alpha BB}$$

$$\left(\tfrac{3}{1}\right) = \frac{\alpha B}{\beta};\ \left(\tfrac{3}{2}\right) = B;\ \left(\tfrac{3}{3}\right) = \gamma$$

$$\left(\tfrac{2}{1}\right) = \frac{\alpha B}{\gamma};\ \left(\tfrac{2}{2}\right) = \frac{\alpha BB}{\gamma A}$$

$$\left(\tfrac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

Scholion.

396. Has determinationes quousque libuerit, continuare licet, in quibus praecipue notari debent casus novas transcendentium species introducentes; quorum primus occurrit si $n = 3$, estque $\left(\tfrac{1}{1}\right) = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$, cujus valorem per productum infinitum supra vidimus esse

$$= \tfrac{3}{1} \cdot \frac{2.6}{4.4} \cdot \frac{5.9}{7.7} \cdot \frac{8.12}{10.10} \text{ etc.}$$

quod ex formula $\left(\tfrac{1}{1}\right)$, ob $n = 3$, etiam est

$$\frac{2}{1.1} \cdot \frac{3.5}{4.4} \cdot \frac{6.8}{7.7} \cdot \frac{9.11}{10.10} \cdot \frac{12.14}{13.13} \text{ etc.}$$

Deinde ex classe $n = 4$ nascitur haec nova forma transcendens:

$$\left(\tfrac{2}{1}\right) = \int \frac{x\,\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt{(1-x^4)}},$$

quae aequatur huic producto infinito

$$\frac{3}{1.2}\cdot\frac{4.7}{5.6}\cdot\frac{8.11}{9.10}\cdot\frac{12.15}{13.14}\cdot\frac{16.19}{17.18}\text{ etc.} = \frac{3}{2}\cdot\frac{2.7}{5.3}\cdot\frac{4.11}{9.5}\cdot\frac{6.15}{13.7}\cdot\frac{8.19}{17.9}\text{ etc}$$

Ex classe $n = 5$ impetramus duas novas formulas transcendentes

$$\left(\tfrac{3}{1}\right) = \int\frac{x^2\partial x}{\sqrt[5]{(1-x^5)^4}} = \int\frac{\partial x}{\sqrt[5]{(1-x^5)^2}} = \frac{4}{1.3}\cdot\frac{5.9}{6.8}\cdot\frac{10.14}{11.13}\cdot\frac{15.19}{16.18}\text{ etc. e}$$

$$\left(\tfrac{2}{2}\right) = \int\frac{x\,\partial x}{\sqrt[5]{(1-x^5)^3}} = \frac{4}{2.2}\cdot\frac{5.9}{7.7}\cdot\frac{10.14}{12.12}\cdot\frac{15.19}{17.17}\text{ etc.}$$

ita ut sit

$$\left(\tfrac{3}{1}\right):\left(\tfrac{2}{2}\right) = \frac{2.2}{1.3}\cdot\frac{7.7}{6.8}\cdot\frac{12.12}{11.13}\cdot\frac{17.17}{16.18}\text{ etc.}$$

Classis $n = 6$ has duas formulas transcendentes suppeditat:

$$1.\ \left(\tfrac{4}{1}\right) = \int\frac{x^3\partial x}{\sqrt[6]{(1-x^6)^5}} = \int\frac{\partial x}{\sqrt[3]{(1-x^6)}} = \tfrac{1}{2}\int\frac{y\,\partial y}{\sqrt[6]{(1-y^3)^5}}$$

$$2.\ \left(\tfrac{3}{2}\right) = \int\frac{x^2\partial x}{\sqrt[3]{(1-x^6)^2}} = \int\frac{x\,\partial x}{\sqrt{(1-x^6)}} = \tfrac{1}{2}\int\frac{\partial y}{\sqrt{(1-y^3)}} = \tfrac{1}{3}\int\frac{\partial z}{\sqrt[3]{(1-zz)}}$$

sumto $y = xx$ et $z = x^3$. Notandum autem est inter has e primam $\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = 2\int\frac{y\,\partial y}{\sqrt[6]{(1-y^6)^4}} = 2\left(\tfrac{2}{2}\right)$ relationem dari, quae est $2\gamma\left(\tfrac{4}{1}\right)\left(\tfrac{2}{2}\right) = \alpha\left(\tfrac{3}{2}\right)\left(\tfrac{3}{2}\right)$, ita ut prima admissa, hic altera sufficiat.