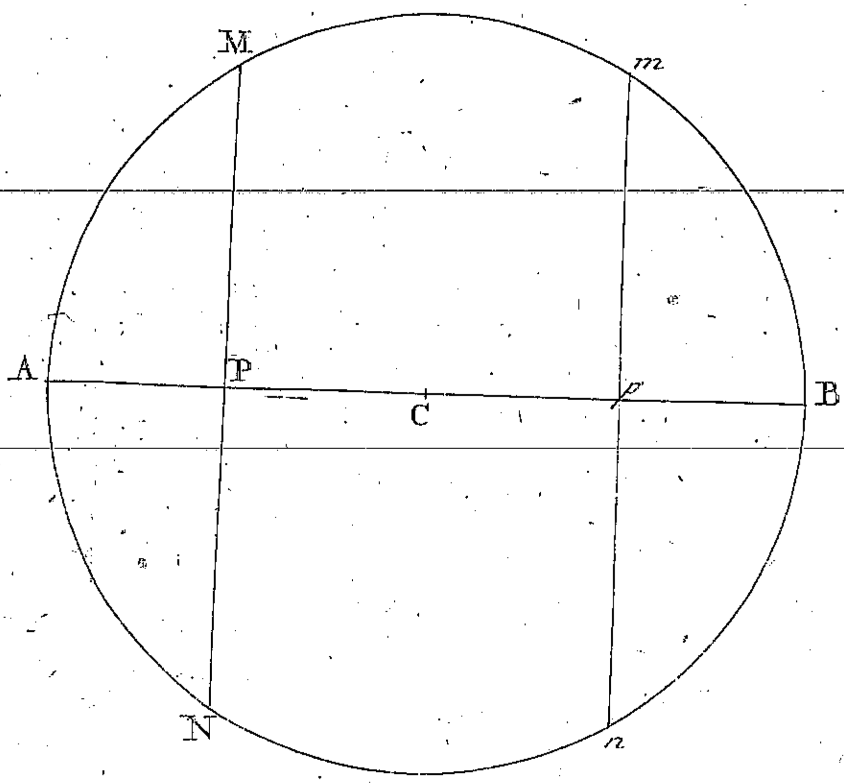
DE SVMMIS SERIERVM RECIPROCARVM.

AVCTORE
Leonh. Eulero.

§. 1.

Tantopere iam pertractatae et inueſtigatae ſunt ſeries reciprocae poteſtatum numerorum naturalium, vt vix probabile videatur de iis noui quicquam inueniri poſſe. Quicunque enim de ſummis ſerierum meditati ſunt, ii fere omnes quoque in ſummas huiusmodi ſerierum inquiſiuerunt, neque tamen vlla methodo eas idoneo modo exprimere potuerunt. Ego etiam iam ſaepius, cum varias ſummandi methodos tradidiſſem, has ſeries diligenter ſum perſecutus, neque tamen quicquam aliud ſum aſſecutus, niſi vt earum ſummam vel proxime veram definiuerim vel ad quadraturas curuarum maxime tranſcendentium reduxerim; quorum illud in diſſertatione proxime praelecta, hoc vero in praecedentibus praeſtiti. Loquor hic autem de ſeriebus fractionum, quarum numeratores ſunt 1, denominatores vero vel quadrata, vel cubi, vel aliae dignitates numerorum naturalium; cuius modi ſunt $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+$ etc., item $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+$ etc. atque ſimiles ſuperiorum poteſtatum, quarum termini generales continentur in hac forma $\frac{1}{x^n}$. Tabula VII.


M
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§. 2. Deductus ſum autem nuper omnino inopinato ad elegantem ſummae huius ſeriei $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc. expreſſionem, quae a circuli quadratura pendet, ita, vt ſi huius ſeriei vera ſumma haberetur, inde ſimul circuli quadratura ſequeretur. Inueni enim ſummae huius ſeriei ſextuplum aequale eſſe quadrato peripheriae circuli, cuius diameter eſt 1; ſeu poſita iſtius ſeriei ſumma $= s$, tenebit $\sqrt{6s}$ ad 1 rationem peripheriae ad diametrum. Huius autem ſeriei ſummam nuper oſtendi proxime eſſe 1, 644934066848226436 4, ex cuius numeri ſextuplo, ſi extrahatur radix quadrata, reipſa prodit numerus 3, 14159265358979323 8 exprimens circuli peripheriam, cuius diameter eſt 1. Iisdam porro veſtigiis, quibus hanc ſummam ſum conſecutus, incedens huius ſeriei $1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} +$ etc. ſummam quoque a quadratura circuli pendere deprehendi. Summa nempe eius per 90 multiplicata dat biquadratum peripheriae circuli, cuius diameter eſt 1. Atque ſimili ratione etiam ſequentium ſerierum, in quibus exponentes dignitatum ſunt numeri pares, ſummas determinare potui.

§. 3. Quo igitur, quemadmodum haec ſum adeptus, commodiſſime oſtendam, totam rem, quo ipſe vſus ſum, ordine exponam. In circulo AMBNA centro C radio AC vel BC $= 1$ deſcripto contemplatus ſum arcum quemcunque AM, cuius ſinus eſt MP, coſinus vero CP. Poſito nunc arcu AM $= s$, ſinu PM $= y$, et coſinu CP $= x$, per methodum iam ſatis cognitam tam ſinus y quam coſinus x ex dato arcu s per ſeries poſſunt

Figura 1.

ſunt definiri, eſt enim, vti paſſim videre licet $y = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} +$ etc. atque $x = 1 - \frac{s^2}{1.2} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2.3.4.5.6} +$ etc. Ex harum ſcilicet aequationum conſideratione ad ſummas ſupra memoratarum ſerierum reciprocarum perueni; quarum aequationum quidem vtraque ad eundem fere ſcopum dirigitur, et hanc ob rem ſufficiet alteram tantum eo, quem ſum expoſiturus, modo tractaſſe.

§. 4. Aequatio ergo prior $y = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} +$ etc. exprimit relationem inter arcum et ſinum. Quare ex ea tam ex dato arcu eius ſinus, quam ex dato ſinu eius arcus determinari poterit. Conſidero autem ſinum y tanquam datum, et inueſtigo, quemadmodum arcum s ex y erui oporteat. Hic vero ante omnia animaduertendum eſt, eidem ſinui y innumerabiles arcus reſpondere, quos ergo innumerabiles arcus aequatio propoſita praebere debebit. Si quidem in iſta aequatione s tanquam incognita ſpectetur, ea infinitas habet dimenſiones, ideoque mirum non eſt, ſi iſta aequatio innumeros contineat factores ſimplices, quorum quisque nihilo aequalis poſitus, idoneum pro s valorem dare debet.

§. 5. Quemadmodum autem, ſi omnes factores huius aequationis cogniti eſſent, omnes quoque radices illius ſeu valores ipſius s innoteſcerent, ita viciſſim ſi omnes valores ipſius s aſſignari poterunt, tum quoque ipſi factores omnes habebuntur. Quo autem eo melius tam de radicibus quam de factoribus iudicare queam, transmuto aequationem propositam in hanc formam: $0 = 1 - \frac{s}{y} + \frac{s^3}{1.2.3.y} - \frac{s^5}{1.2.3.4.5.y} +$ etc. Si nunc omnes radices huius aequationis seu omnes arcus, quorum idem est sinus y, fuerint A, B, C, D, E etc. tum factores quoque erunt omnes istae quantitates, $1 - \frac{s}{A}$, $1 - \frac{s}{B}$, $1 - \frac{s}{C}$, $1 - \frac{s}{D}$ etc. Quamobrem erit $1 - \frac{s}{y} + \frac{s^3}{1.2.3.y} - \frac{s^5}{1.2.3.4.5.y} +$ etc. $= (1 - \frac{s}{A})(1 - \frac{s}{B})(1 - \frac{s}{C})(1 - \frac{s}{D})$ etc.

§. 6. Ex natura autem et resolutione aequationum constat, esse coëfficientem termini, in quo inest s, seu $\frac{1}{y}$ aequalem summae omnium coëfficientium ipsius s in factoribus seu $\frac{1}{y} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} +$ etc. Deinde est coëfficiens ipsius s^2, qui est $= 0$, ob hunc terminum in aequatione deficientem, aequalis aggregato factorum ex binis terminis seriei, $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D}$ etc. Porro erit $-\frac{1}{1.2.3.y}$ aequale aggregato factorum ex quaternis terminis eiusdem seriei $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D}$ etc. Similique modo erit $0 =$ aggregato factorum ex quaternis terminis eiusdem seriei, et $+\frac{1}{1.2.3.4.5.y} =$ aggregato factorum ex quinis terminis istius seriei, et ita porro.

§. 7. Posito autem minimo arcu $AM = A$, cuius sinus est $PM = y$, et semiperipheria circuli $= p$, erunt $A, p-A, 2p+A, 3p-A, 4p+A, 5p-A, 6p+A$ etc. item $-p-A, -2p+A, -3p-A, -4p+A, -5p-A$, etc. omnes arcus, quorum sinus est idem y. Quam igitur ante assumsimus seriem $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D}$, etc. ea transmutatur in hanc $\frac{1}{A}, \frac{1}{p-A}, \frac{1}{-p-A}, \frac{1}{2p+A}, \frac{1}{-2p+A}, \frac{1}{3p-A}, \frac{1}{-3p-A}, \frac{1}{4p+A}, \frac{1}{-4p+A}$ etc. Horum ergo omnium termi-

terminorum ſumma eſt $= \frac{1}{y}$; ſumma autem factorum ex binis terminis huius ſeriei eſt aequalis 0; ſumma factorum ex ternis $= \frac{-1}{1.2.3.y}$, ſumma factorum ex quaternis $= 0$; ſumma factorum ex quinis $= \frac{+1}{1.2.3.4.5.y}$; ſumma factorum ex ſenis $= 0$. Atque ita porro.

§. 8. Si autem habeatur ſeries quaecunque $a + b + c + d + e + f +$ etc. cuius ſumma ſit α, ſumma factorum ex binis terminis $= \beta$; ſumma factorum ex ternis $= \gamma$; ſumma factorum ex quaternis $= \delta$, etc. erit ſumma quadratorum ſingulorum terminorum, hoc eſt $a^2 + b^2 + c^2 + d^2 +$ etc. $= \alpha^2 - 2\beta$; ſumma vero cuborum $a^3 + b^3 + c^3 + d^3$ etc. $= \alpha^3 - 3\alpha\beta + 3\gamma$; ſumma biquadratorum $= \alpha^4 - 4\alpha^2\beta + 4\alpha\gamma + 2\beta^2 - 4\delta$. Quo autem clarius appareat, quomodo hae formulae progrediantur, ponamus ipſorum terminorum a, b, c, d, etc. ſummam eſſe $= P$, ſummam quadratorum $= Q$, ſummam cuborum $= R$, ſummam biquadratorum $= S$, ſummam poteſtatum quintarum $= T$, ſummam ſextarum $= V$ etc. His poſitis erit $P = \alpha$; $Q = P\alpha - 2\beta$; $R = Q\alpha - P\beta + 3\gamma$; $S = R\alpha - Q\beta + P\gamma + 4\delta$; $T = S\alpha - R\beta + Q\gamma - P\delta + 5\varepsilon$; etc.

§. 9. Cum igitur in noſtro caſu ſeriei $\frac{1}{A}$, $\frac{1}{p - A}$, $\frac{1}{-p - A}$, $\frac{1}{2p + A}$, $\frac{1}{-2p + A}$, $\frac{1}{3p - A}$, $\frac{1}{-3p - A}$ etc. ſumma omnium terminorum ſeu α ſit $= \frac{1}{y}$; ſumma factorum ex binis ſeu $\beta = 0$, atque vlterius $\gamma = \frac{-1}{1.2.3.y}$; $\delta = 0$; $\varepsilon = \frac{+1}{1.2.3.4.5.y}$, $\zeta = 0$; etc. erit ſumma ipſorum illorum terminorum $P = \frac{1}{y}$; ſumma quadratorum illorum terminorum $Q = \frac{P}{y} = \frac{1}{y^2}$;

ſumma

ſumma cuborum illorum terminorum $R = \frac{Q}{y} - \frac{1}{1.2.y}$; ſumma biquadratorum $S = \frac{R}{y} - \frac{P}{1.2.3.y}$. Atque porro $T = \frac{S}{y} - \frac{Q}{1.2.3.y} + \frac{1}{1.2.3.4.y}$; $V = \frac{T}{y} - \frac{R}{1.2.3.y} + \frac{P}{1.2.3.4.5.y}$; $W = \frac{V}{y} - \frac{S}{1.2.3.y} + \frac{Q}{1.2.3.4.5y} - \frac{1}{1.2.3.4.5.6\,y}$. Ex qua lege facile reliquarum altiorum poteſtatum ſummae determinantur.

§. 10. Ponamus nunc ſinum $PM = y$ aequalem radio, vt ſit $y = 1$, erit minimus arcus A cuius ſinus eſt 1 quarta peripheriae pars, $= \frac{1}{2}p$, ſeu denotante q quartam peripheriae partem erit $A = q$ et $p = 2q$. Superior ergo ſeries abibit in iſtam $\frac{1}{q}, \frac{1}{q}, -\frac{1}{3q}, -\frac{1}{3q}, +\frac{1}{5q}, +\frac{1}{5q}, -\frac{1}{7q}, -\frac{1}{7q}, +\frac{1}{9q}, +\frac{1}{9q}$, etc. binis terminis exiſtentibus aequalibus. Horum ergo terminorum ſumma, quae eſt $\frac{2}{q}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.})$ aequalis eſt ipſi $P = 1$. Hinc igitur oritur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} = \frac{q}{2} = \frac{p}{4}$. Huius ergo ſeriei quadruplum aequatur ſemiperipheriae circuli, cuius radius eſt 1, ſeu toti peripheriae circuli, cuius diameter eſt 1. Atque haec eſt ipſa ſeries a *Leibnitio* iam pridem prolata, qua circuli quadraturam definiuit. Ex quo magnum huius methodi, ſi cui forte ea non ſatis certa videatur, firmamentum elucet; ita vt de reliquis, quae ex hac methodo deriuabantur, omnino non liceat dubitari.

§. 11. Sumamus nunc inuentorum terminorum pro caſu quo $y = 1$, quadrata, prodibitque haec ſeries $+\frac{1}{q^2} + \frac{1}{q^2} + \frac{1}{9q^2} + \frac{1}{9q^2} + \frac{1}{25q^2} + \frac{1}{25q^2} + \text{etc.}$ cuius ſumma eſt $\frac{2}{q^2}(\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.})$, quae ergo aequalis eſſe debet ipſi $Q = P = 1$. Ex quo ſequitur huius ſeriei $1 + \frac{1}{9}$

+

$+\frac{1}{25}+\frac{1}{49}+$ etc. ſummam eſſe $=\frac{q^2}{2}=\frac{p^2}{8}$; denotante p totam circuli peripheriam, cuius diameter eſt $=1$. Summa autem huius ſeriei $1+\frac{1}{9}+\frac{1}{25}+$ etc. pendet a ſumma ſeriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+$ etc. quia haec quarta ſui parte minuta illam dat. Eſt ergo ſumma huius ſeriei aequalis ſummae illius cum ſui triente. Quamobrem erit $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+$ etc. $=\frac{p^2}{6}$, ideoque huius ſeriei ſumma per 6 multiplicata aequalis eſt quadrato peripheriae circuli cuius diameter eſt 1; quae eſt ipſa propoſitio cuius initio mentionem feci.

§. 12. Cum igitur caſu quo $y=1$, ſit $P=1$ et $Q=1$, erunt reliquarum litterarum R, S, T, V etc. vt ſequitur: $R=\frac{1}{2}$; $S=\frac{1}{3}$; $T=\frac{5}{24}$; $V=\frac{2}{15}$; $W=\frac{61}{720}$; $X=\frac{17}{315}$ etc. Cum autem ſumma cuborum ipſi $R=\frac{1}{2}$ ſit aequalis, erit $\frac{2}{q^3}(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.})=\frac{1}{2}$. Quare erit $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.}=\frac{q^3}{4}=\frac{p^3}{32}$. Huius ideo ſeriei ſumma per 32 multiplicata dat cubum peripheriae circuli cuius diameter eſt 1. Simili modo ſumma biquadratorum, quae eſt $\frac{2}{p^4}(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+$ etc.) aequalis eſſe debet $\frac{1}{3}$, ideoque erit $1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.}=\frac{q^4}{6}=\frac{p^4}{96}$. Eſt vero haec ſeries per $\frac{16}{15}$ multiplicata aequalis huic $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{6^4}+$ etc. quare iſta ſeries aequalis eſt $\frac{p^4}{90}$; ſeu ſeriei $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+$ etc. ſumma per 90 multiplicata dat biquadratum peripheriae circuli cuius diameter eſt 1.

§. 13. Simili modo inuenientur ſummae ſuperiorum poteſtatum; prodibit autem vt ſequitur $1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=\frac{5q^5}{48}=\frac{5p^5}{1536}$; atque $1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}$

$+$ etc. $= \frac{q^6}{15} = \frac{p^6}{960}$. Inuenta vero huius feriei fumma, cognofcetur fimul fumma huius feriei $1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} +$ etc. quae erit $= \frac{p^6}{945}$. Porro pro poteftatibus feptimis erit $1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} -$ etc. $= \frac{61 q^7}{1440} = \frac{61 p^7}{184320}$ ac pro octauis $1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} +$ etc. $= \frac{17 q^8}{630} = \frac{17 p^8}{161280}$; vnde deducitur $1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} +$ etc. $= \frac{p^8}{9450}$. Obferuandum autem eft de his feriebus in potentiis exponentium imparium figna terminorum alternari, pro poteftatibus paribus vero effe aequalia; hocque in caufa eft, quod huius generalis feriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ etc. iis tantum cafibus fumma poffit exhiberi, quibus n eft numerus par. Praeterea quoque notandum eft, fi feriei $1, 1, \frac{1}{2}, \frac{1}{3}, \frac{5}{24}, \frac{2}{15}, \frac{61}{720}, \frac{17}{315}$ etc. quos valores pro litteris P, Q, R, S etc. inuenimus, terminus generalis poffet affignari, tum eo ipfo quadraturam circuli exhibitum iri.

§. 14. In his pofuimus finum PM aequalem radio, videamus ergo quales feries prodeant, fi ipfi y alii valores tribuantur. Sit igitur $y = \frac{1}{\sqrt{2}}$, cui finui minimus arcus refpondens eft $\frac{1}{4} p$. Pofito ergo $A = \frac{1}{4} p$ erit feries terminorum fimplicium feu primae poteftatis ifta $\frac{4}{p} + \frac{4}{3p} - \frac{4}{5p} - \frac{4}{7p} + \frac{4}{9p} + \frac{4}{11p} -$ etc. cuius feriei fumma P aequalis eft $\frac{1}{y} = \sqrt{2}$. Habebitur ergo $\frac{p}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15}$ etc. quae feries tantum ratione fignorum a *Leibnitiana* differt, et a *Newtono* iam dudum eft prolata. Summa vero quadratorum illorum terminorum nempe $\frac{16}{p^2} (1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} +$ etc.$)$ aequalis eft ipfi $Q = 2$. Erit

ergo

ergo $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} +$ etc. $= \frac{p^2}{8}$, vti ante eſt inuentum.

§. 3. Si fiat $y = \frac{\sqrt{3}}{2}$ erit minimus arcus huic ſinui reſpondens $60°$, ideoque $A = \frac{1}{3}p$. Hoc ergo caſu ſequens prodibit ſeries terminorum $\frac{3}{p} + \frac{3}{2p} - \frac{3}{4p} - \frac{3}{5p} + \frac{3}{7p} + \frac{3}{8p}$ etc. quorum terminorum ſumma aequalis eſt ipſi $\frac{1}{y} = \frac{2}{\sqrt{3}}$. Habebitur ergo $\frac{2p}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} - \frac{1}{10} - \frac{1}{11} +$ etc. Summa vero quadratorum illorum terminorum eſt $= \frac{1}{y^2} = \frac{4}{3}$; vnde ſequitur fore $\frac{4p^2}{27} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{25} + \frac{1}{49} + \frac{1}{64} +$ etc. in qua ſerie deſunt termini ternario conſtantes. Pendet autem haec ſeries quoque ab iſta $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ etc. cuius ſumma erat inuenta $= \frac{p^2}{6}$; nam ſi haec ſeries ſui parte nona minuatur prodit ipſa ſuperior ſeries, cuius ideo ſumma debet eſſe $= \frac{p^2}{6}(1 - \frac{1}{9}) = \frac{4pp}{27}$. Simili modo ſi alii aſſumantur ſinus, aliae prodibunt ſeries, tam ſimplicium, quam terminorum quadratorum altiorumque poteſtatum, quarum ſummae quadraturam circuli inuoluent.

§. 16. At ſi ponatur $y = 0$, huiusmodi ſeries non amplius aſſignari poterunt, propter y in denominatorem poſitum, ſeu aequationem initialem per y diuiſam. Alio autem modo ſeries inde deduci poterunt, quae cum ſint ipſae ſeries $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ etc. ſi n eſt numerus par: quemadmodum harum ſerierum ſummae ſint inueniendae, ſeorſum ex hoc caſu quo $y = 0$ deducam. Poſito vero $y = 0$ ipſa aequatio fundamentalis abit in hanc $0 = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} +$ etc. cuius aequationis radices dant omnes arcus, quorum ſinus eſt $= 0$.

Eſt autem vna minimaque radix $s=0$, quare aequatio per s diuiſa exhibebit reliquos arcus omnes, quorum ſinus eſt $=0$, qui arcus proinde erunt radices huius aequationis $0=1-\frac{s^2}{1.2.3}+\frac{s^4}{1.2.3.4.5}-\frac{s^6}{1.2.3.4.5.6.7}+$ etc. Ipſi vero arcus quorum ſinus eſt $=0$ ſunt $p, -p, +2p, -2p, 3p, -3p$ etc. quorum binorum alter alterius eſt negatiuus, id quod quoque ipſa aequatio propter dimenſiones ipſius s tantum pares indicat. Quare diuiſores illius aequationis erunt $1-\frac{s}{p}$, $1+\frac{s}{p}$, $1-\frac{s}{2p}$, $1+\frac{s}{2p}$, etc. atque coniungendis binis horum diuiſorum erit $1-\frac{s^2}{1.2.3}+\frac{s^4}{1.2.3.4.5}-\frac{s^6}{1.2.3.4.5.6.7}+\text{etc.}=(1-\frac{s^2}{p^2})(1-\frac{s^2}{4p^2})(1-\frac{s^2}{9p^2})(1-\frac{s^2}{16p^2})$ etc.

§. 17. Manifeſtum iam eſt ex natura aequationum, fore coefficientem ipſius ss ſeu $\frac{1}{1.2.3}$ aequalem $\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+$ etc. Summa vero factorum ex binis terminis huius ſeriei erit $=\frac{1}{1.2.3.4.5}$; ſummaque factorum ex ternis $=\frac{1}{1.2.3.4.5.6.7}$ etc. Hanc ob rem erit iuxta §. 8. $\alpha=\frac{1}{1.2.3}$; $\beta=\frac{1}{1.2.3.4.5}$; $\gamma=\frac{1}{1.2.3.4.5.6.7}$; etc. atque poſita quoque ſumma terminorum $\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+$ etc. $=P$, et ſumma quadratorum eorundem terminorum $=Q$; ſumma cuborum $=R$; ſumma biquadratorum $=S$; etc. erit per §. 8. $P=\alpha=\frac{1}{1.2.3}=\frac{1}{6}$; $Q=P\alpha-2\beta=\frac{1}{90}$; $R=Q\alpha-P\beta+3\gamma=\frac{1}{945}$; $S=R\alpha-Q\beta+P\gamma-4\delta=\frac{1}{9450}$; $T=S\alpha-R\beta+Q\gamma-P\delta+5\varepsilon=\frac{1}{93555}$; $V=T\alpha-S\beta+R\gamma-Q\delta+P\varepsilon-6\zeta=\frac{691}{6825.93555}$ etc.

§. 18.

§. 18. Ex his ergo deriuantur ſummae ſequentes:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \text{ etc.} = \frac{p^2}{6} = P$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} \text{ etc.} = \frac{p^4}{90} = Q$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} \text{ etc.} = \frac{p^6}{945} = R$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} \text{ etc.} = \frac{p^8}{9450} = S$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} \text{ etc.} = \frac{p^{10}}{93555} = T$$

$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} \text{ etc.} = \frac{691 p^{12}}{6825 \cdot 93555} = V.$$

quae ſeries ex data lege attamen multo labore ad altiores poteſtates produci poſſunt. Diuidendis autem ſingulis ſeriebus per praecedentes orientur ſequentes aequationes: $p^2 = 6P = \frac{15Q}{P} = \frac{21R}{2Q} = \frac{10S}{R} = \frac{99T}{10S} = \frac{6825V}{691T}$ etc. quibus expreſſionibus ſingulis quadratum peripheriae cuius diameter eſt 1, aequatur.

§. 19. Cum autem harum ſerierum ſummae etiamſi vero proxime facile exhiberi poſſent, tamen non multum adiumenti afferre queant ad peripheriam circuli vero proxime exprimendam propter radicem quadratam, quae extrahi deberet; ex prioribus ſeriebus eliciemus expreſſiones, quae ipſi peripheriae p ſint aequales. Prodibit autem vt ſequitur:

$$p = 4\ \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right)$$

$$p = 2\ \left(\frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \text{etc.}}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}}\right)$$

$$p = 4\ \left(\frac{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.}}{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \text{etc.}}\right)$$

$$p = 3\ \left(\frac{1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \frac{1}{11^4} + \text{etc.}}{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.}}\right)$$

$$p = \tfrac{16}{5}\ \left(\frac{1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \text{etc.}}{1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \frac{1}{11^4} + \text{etc.}}\right)$$

$$p = \tfrac{25}{8}\ \left(\frac{1 + \frac{1}{3^5} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \frac{1}{11^6} + \text{etc.}}{1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \text{etc.}}\right)$$

$$p = \tfrac{192}{61}\ \left(\frac{1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \text{etc.}}{1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \frac{1}{11^6} + \text{etc.}}\right)$$

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